Computer Graphics II
2: Repetition CG-I & Foundations

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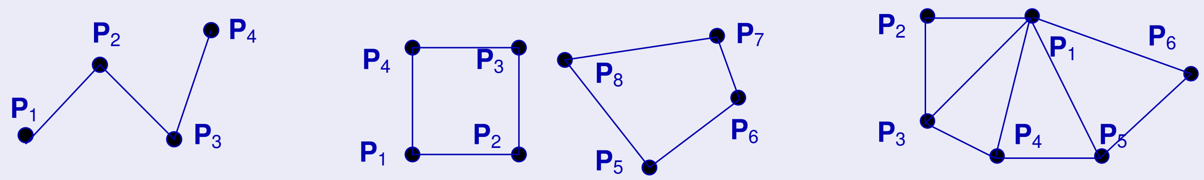
Structure of this Chapter & Motivation

Structure of Chapter

- *Subsection 1: Essential foundations handled in CG-I like affine spaces*
- *Subsection 2: Basic concepts about (surface-like) geometry*

Motivation

Modeling so far: *Purely polygonal objects (meshes)*



Evaluation:

+ *Able to generate arbitrary shapes*
– *Geometry manipulation only at the finest level (vertices)*
– *No smooth surfaces*

Rendering *is almost always based on polygons.*

Repetition: Affine Space

Notation (Affine Space)

All geometric objecte are defined using points (vertices) represented in an affine space*.*

Affine Space A: *Vector space extended by points.*
Given vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in A$, points $\mathbf{P}, \mathbf{Q} \in A$ and $a, b \in \mathbb{R}$, we have:

- $\mathbf{P} - \mathbf{Q}$ *is a vector*
- $\mathbf{P} + \vec{\mathbf{v}}, \mathbf{P} + a\vec{\mathbf{v}}$ *are points*

A coordinate system of A *consists of a vector basis $\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_n\}$ of A and an origin $\mathcal{O}$.*
Coordinate representation of $\mathbf{P} \in A$ using the position vector $\vec{\mathbf{p}}$ of $\mathbf{P}$:

$$\mathbf{P} = \mathcal{O} + \vec{\mathbf{p}} = \mathcal{O} + p_1\vec{\mathbf{u}}_1 + p_2\vec{\mathbf{u}}_2 + \ldots + p_n\vec{\mathbf{u}}_n, \quad \mathbf{P} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

Affine Space

Notation (Affine Combination and Convex Hull)

Affine Combination: *Given points $\mathbf{P}_1, \ldots, \mathbf{P}_k \in A$ and scalar values $s_1, \ldots, s_k$ with $\sum_{i=1}^{k} s_i = 1$ (partition of unity), $\sum_{i=1}^{k} s_i\mathbf{P}_i \in A$ again is a point, since*

$$\sum_{i=1}^{k} s_i\mathbf{P}_i = s_1\mathbf{P}_1 + s_2\mathbf{P}_2 + \ldots + s_k\mathbf{P}_k$$
$$= \underbrace{(s_1 + \ldots + s_k)}_{=1}\mathbf{P}_1 + s_2\underbrace{(\mathbf{P}_2 - \mathbf{P}_1)}_{\vec{\mathbf{v}}_2} + \ldots s_k\underbrace{(\mathbf{P}_k - \mathbf{P}_1)}_{\vec{\mathbf{v}}_k}$$

Convex Hull $\mathcal{H}$ *of a set of points $\mathbf{P}_1, \ldots, \mathbf{P}_k \in A$ is the* smallest convex set *including them. $\mathcal{H}$ contains all*

$$\mathbf{Q} = \sum_{i=1}^{k} s_i\mathbf{P}_i, \text{ with } \sum_{i=1}^{k} s_i = 1 \text{ and } s_i \geq 0 \ \forall i = 1, \ldots, k$$

Affine Combination

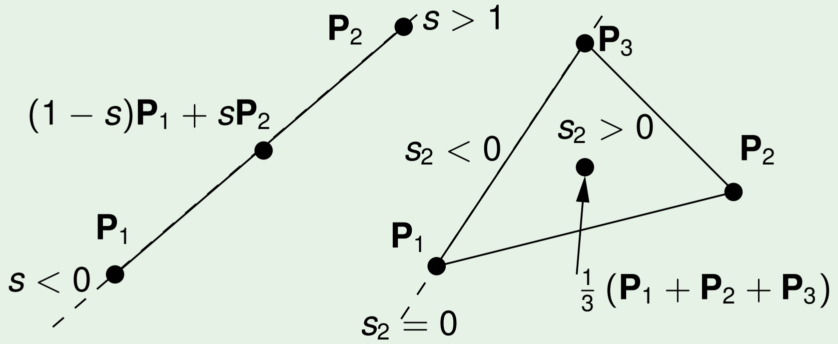
Example (Affine Combination)

Line passing through points $\mathbf{P}_1 \neq \mathbf{P}_2$:
$G : \mathbf{P}(s) = \mathbf{P}_1 + s\,(\mathbf{P}_2 - \mathbf{P}_1) = (1-s)\mathbf{P}_1 + s\mathbf{P}_2,\ s \in \mathbb{R}$
The mapping $s \to \mathbf{P}(s)$ is bijectiv, the convex hull is the line segment $\overline{\mathbf{P}_1\mathbf{P}_2}$

Plane passing through three non collinear points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$

$$E : \mathbf{P}_1 + s_1\,(\mathbf{P}_2 - \mathbf{P}_1) + s_2\,(\mathbf{P}_3 - \mathbf{P}_1) = (1 - s_1 - s_2)\mathbf{P}_1 + s_1\mathbf{P}_2 + s_2\mathbf{P}_3,\ s_1, s_2 \in \mathbb{R}$$

The mapping $(s_1, s_2) \to \mathbf{P}(s_1, s_2)$ is bijectiv, the convex hull is the triangle $\Delta\,(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$.



Baryzentric Coordinates

Notation (Baryzentric Coordinates)

General Question: *How to compute affine weights for a given point* $\mathbf{P}$ *with respect to* $\mathbf{P}_i \in A,\ i = 0, \ldots, k$

Affine Independence: $\mathbf{P}_i \in A,\ i = 0, \ldots, k$ *with* $\dim(A) \geq k$ *are affine independent, if and only if*

The vectors $\vec{\mathbf{v}}_i = \mathbf{P}_i - \mathbf{P}_1,\ i = 1, \ldots, k$ *are linear independent*

$$\textit{respectively} \sum_{i=0}^{k} s_i\mathbf{P}_i = \sum_{i=0}^{k} t_i\mathbf{P}_i \textit{ with } \sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1 \Leftrightarrow s_i = t_i\ \forall i = 0, \ldots, k$$

Baryzentric Coordinates *are the unique weights* s_i *of points* $\mathbf{P}$ *with respect to the affine independent points* $\mathbf{P}_i,\ k = 1, \ldots, k$

Example (Baryzentric Coordinates for a Triangle in $\mathbb{R}^2$)

Given triangle with vertices $\mathbf{Q}_i,\ i = 0, 1, 2$ and point $\mathbf{P}$

Wanted: Baryzentric coord. $s_i,\ i = 0, 1, 2$ with $\mathbf{P} = \sum_{i=0}^{2} s_i \mathbf{Q}_i,\ \sum_{i=0}^{2} s_i = 1$

Approach: With $s_0 = 1 - s_1 - s_2,\ \vec{\mathbf{u}}_i = \mathbf{Q}_i - \mathbf{Q}_0$ and $\vec{\mathbf{v}} = \mathbf{P} - \mathbf{Q}_0$ we have

$$(\mathbf{Q}_1 - \mathbf{Q}_0, \mathbf{Q}_2 - \mathbf{Q}_0)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \mathbf{P} - \mathbf{Q}_0 \Leftrightarrow \begin{pmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

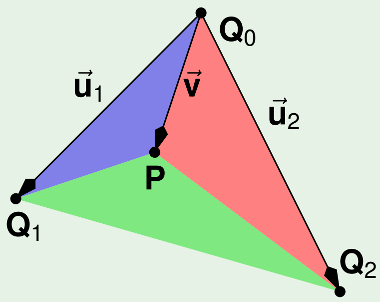
$$\Rightarrow s_1 = \frac{u_{2y}v_x - u_{2x}v_y}{u_{1x}u_{2y} - u_{2x}u_{1y}} \text{ and } s_2 = \frac{u_{1x}v_y - u_{1y}v_x}{u_{1x}u_{2y} - u_{2x}u_{1y}}$$

Geometric Interpretation: s_i resembles an area ratio

$$s_1 = \frac{\text{area}(\Delta(\mathbf{Q}_0, \mathbf{P}, \mathbf{Q}_2))}{\text{area}(\Delta(\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2))}$$

$$s_2 = \frac{\text{area}(\Delta(\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{P}))}{\text{area}(\Delta(\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2))}$$

$$\text{analog } s_0 = \frac{\text{area}(\Delta(\mathbf{P}, \mathbf{Q}_1, \mathbf{Q}_2))}{\text{area}(\Delta(\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2))}$$



Definition (Affine Transformations $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (usually $n = 2, 3$))

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *affine*, if affine combinations are invariant under T, i.e.

$$\text{Given: Points } \mathbf{P}_i, \text{ weights } s_i,\ i = 1, \ldots, k \text{ with } \sum_{i=1}^{k} s_i = 1$$

$$\text{Property: For } \mathbf{P} = \sum_{i=1}^{k} s_i \mathbf{P}_i \text{ we have } T(\mathbf{P}) = T(\sum_{i=1}^{k} s_i \mathbf{P}_i) = \sum_{i=1}^{k} s_i T(\mathbf{P}_i)$$

Notably: T leaves baryzentric coordinates unchanged!

Characterization: T is affin if and only if T is linear and/or a translation

$$T(\mathbf{P}) = M \cdot \mathbf{P} + \vec{\mathbf{t}}, \text{ with } M \in \mathbb{R}^{n \times n},\ \vec{\mathbf{t}} \in \mathbb{R}^n$$

Homogenous Notation for affine transformations

$$\mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \to \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \text{ and } M\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} \to \begin{bmatrix} & M & & t_x \\ & & & t_y \\ & & & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Geomtery Types

Notation

Geomtery Types

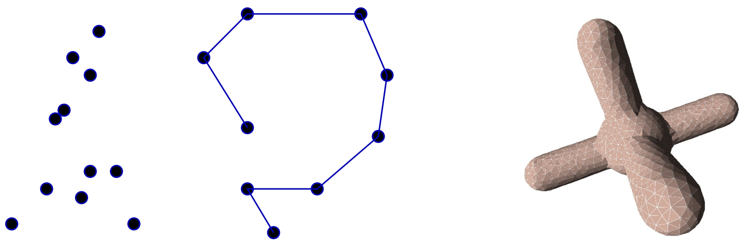
Geometry: *A set of points (mainly in $\mathbb{R}^2$ or or $\mathbb{R}^3$) without specific structure*

Curve: *A geometry with (locally) one dimensional structure.*

Surface: *A geometry with (locally) two dimensional structure.*

Volume: *A geometry with (locally) three dimensional structure.*
Note: Volumentric objects are often represented by their surface.

Model / Object: *A geometry with semantics, e.g. house, car, horse*



Points Curve Surface

Manifold and Orientability

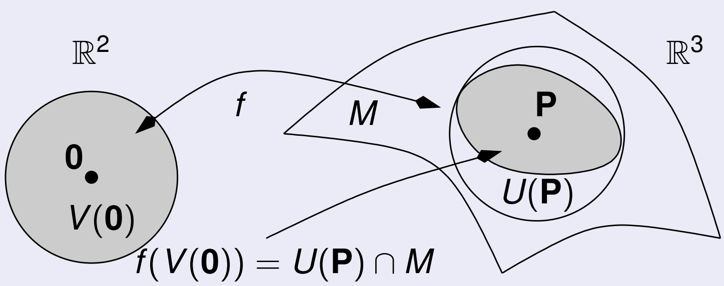
Notation (Manifold and Orientability)

Problem: "Locally *n* dimensional structure" *is a very vage statement*

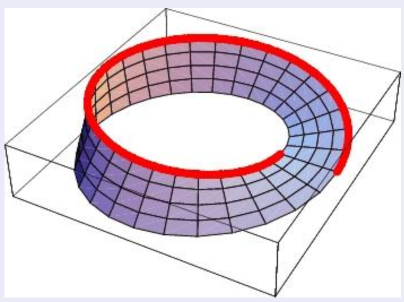
k-Manifold: *A set $M \subset \mathbb{R}^d$ is k-manifold, $k \leq d$, if*

$$\forall \mathbf{P} \in M : \exists \text{ neighborhood } U(\mathbf{P}) \subset \mathbb{R}^d, V(\mathbf{0}) \subset \mathbb{R}^k \text{ of } \mathbf{P} \text{ resp. of } \mathbf{0}$$
$$\text{and a continuous bijective mapping } f : V(\mathbf{0}) \longrightarrow U(\mathbf{P}) \cap M$$

Orientability ($k = 2,\ d = 3$): *A two-sided manifold (surface)*



Example 2-manifold

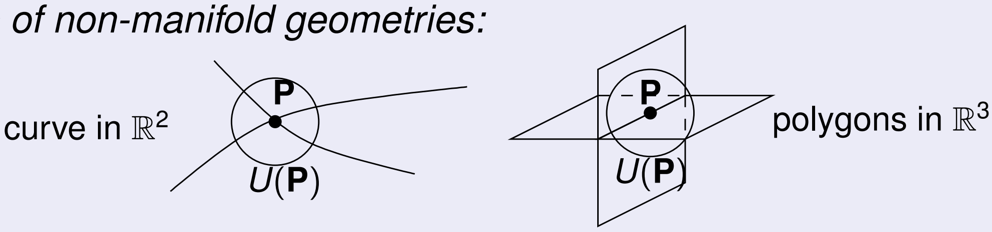


Moebius-Band (single sided)

Manifold and Boundary

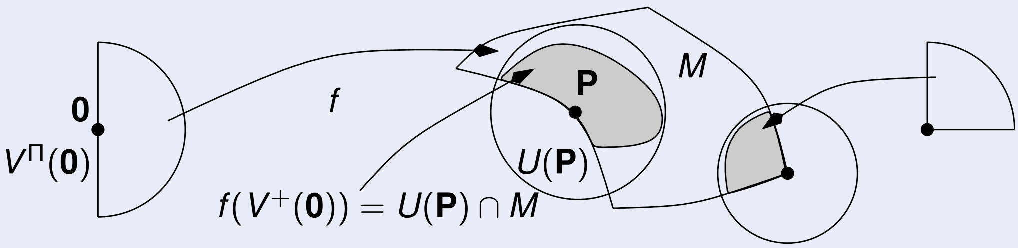
Remark

- *Example of non-manifold geometries:*



- *Manifolds are per se unbounded, i.e. they have no boundary (e.g. sphere or plane)*
- *The boundary of a manifold ∂M is defined via half-open neighborhoods*

$$\Pi \subset \{1, \dots, k\},\ V^{\Pi}(\mathbf{0}) = \{\mathbf{Q} \in V(\mathbf{0}) \ :\ q_i \geq 0 \forall i \in \Pi\} \subset \mathbb{R}^k$$



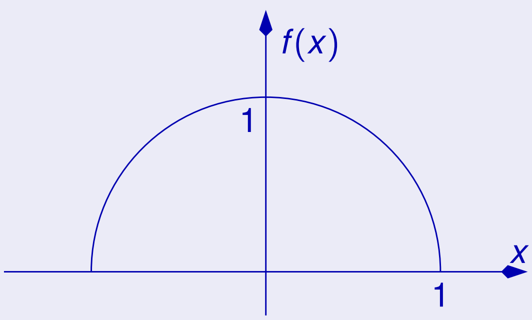
Mathematic Representation of Geometry

Notation

Polygon: *Planar 2-manifold with polygon-line as boundary*

Mesh: *Set of polygons with common edges and vertices*

Parametric, *e.g. functional curve of a half cycle: $f(x) = \sqrt{1 - x^2}$, $x \in [-1, 1]$ as function of x*



Implicit Representation (not handled in CG-II): *Definition of a $n-1$-dim. geometry in an n-dim. space via a scalar function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ ($n = 2$: curve, $n = 3$: surface) with respect to an iso value a:*

$$\text{General:} \quad \{\mathbf{P} = (p_x, p_y, p_z) \in \mathbb{R}^3 \ :\ f(\mathbf{P}) = a\}$$
$$\text{Example sphere:} \quad f(\mathbf{P}) = p_x^2 + p_y^2 + p_z^2,\ a = r^2$$

Subdivision Surfaces: *Recursively subdivision of meshes leading to smooth limit surfaces.*

Notation

Primitive:
- *Basic form for modeling*
- *is based on* **one** *math. representation*
- *Example* sphere*: exact as implicite geometry or approximated by polygons*

Object: *Combination of several primitives, e.g. using boolean operations.*

Modeling Technique: *Method to combine or generate objects out of several primitives and/or parameter values*