# Diagnostic Medical Image Processing

# Reconstruction from X-Ray Projections

## Computerized Tomography

- $\tau o \mu o \sigma$ = tomos = slice
- reconstruction of functions from line integrals:
  - transmission CT: radiology
    - computer tomography: (mostly) diagnostic radiology
    - 3-D angiography: (mostly) interventional radiology
  - emission CT: nuclear medicine
    - PET: positron emission tomography
    - SPECT: single particle emission tomography
  - ultrasound tomography

# X-Ray Attenuation Law

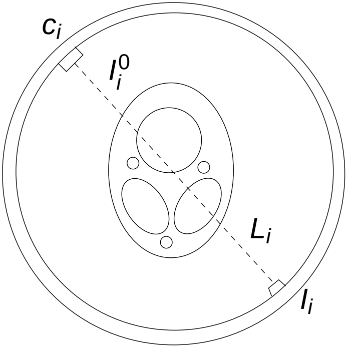



Figure: Acquisition scenario

- X-ray focus $c_i$ of the $i$-th image,
- projection beam $L_i$ of the $i$-th view,
- $I_i^0$ initial intensity, and
- $I_i$ measured intensity

**Core problem in CT:** Compute the original function from line integrals measured by $I_i$, where $c_i$'s are usually on a circle.

# Fan Beam Acquisition Geometry

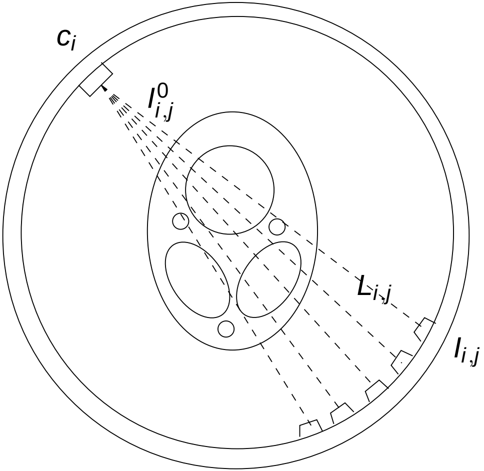



Figure: CT reconstruction: many rays $L_{i,j}$, $j = 1, 2, \ldots$ with single focus $c_i$

# X-Ray Attenuation Law

For the $j$–th X-ray beam the observed intensity of the $i$–th view is

$$I_{i,j} = I^0_{i,j} \exp\left( - \int_{L_{i,j}} f(x(l), y(l))\, dl \right) \quad ,$$

where

- $I^0_{i,j}$ is original intensity (no object),
- $L_{i,j}$ is the $j$-th line of the X-ray beam of the $i$-th view, and
- $f(x(l), y(l))$ is the 2–D function value (density) of X-rayed object at $(x(l), y(l))$.

# System of Equations

For $N_\mathrm{v}$ views ($i = 1, 2, \ldots, N_\mathrm{v}$) and $N_\mathrm{t}$ samples ($j = 1, 2, \ldots, N_\mathrm{t}$) in each view we get the following set of integral equations:

$$\log \frac{I_{i,j}}{I_{i,j}^0} = -\int_{L_{i,j}} f(x(l), y(l))\, dl, \text{ for } i = 1, \ldots, N_\mathrm{v} \text{ and } j = 1, \ldots, N_\mathrm{t}.$$

# Industrial CT

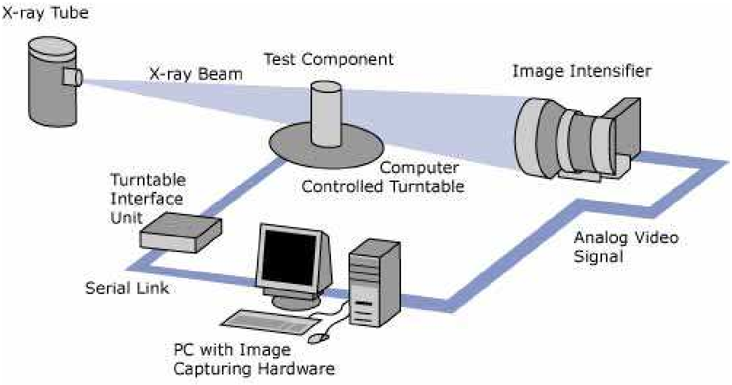
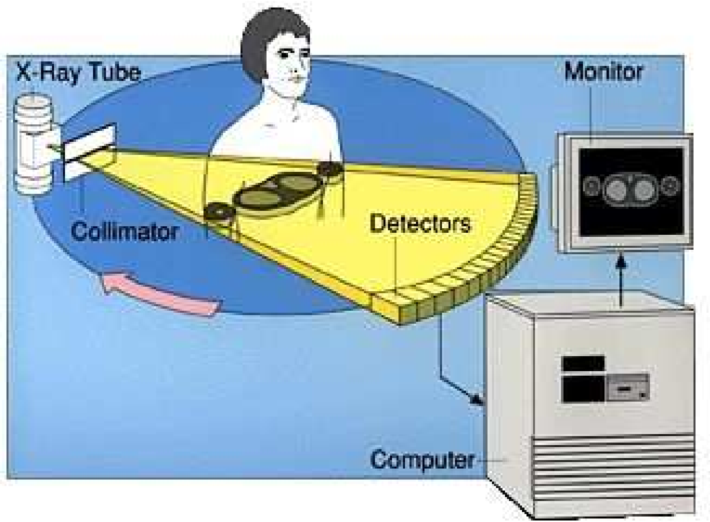



Figure: Principle of Computed Tomography: Rotate Object

# Medical CT



# 3-D Computerized Tomography

Recent developments in

- detector technology,
- X-ray tube technology,
- computational power of computers, and
- algorithms for 3-D reconstruction

are the main driving forces in the innovation of 3-D reconstruction systems since the late 90-ies of the last century.

# Highlight I: 3-D Reconstruction in dual CT

Figure: Dual source CT introduced in 2005 by Siemens Healthcare (image: Siemens Healthcare)

# Highlight II: 3-D Reconstruction using 320 Detector Rows

Figure: November 2007: Toshiba introduces a 320 slice scanner (image: Toshiba)

# Highlight III: 3-D Reconstruction in Dental Medicine

Figure: October 2006: Dental 3D reconstruction system (image: http://www.planmeca.com)

# Highlight IV: 3-D Reconstruction in Angiography

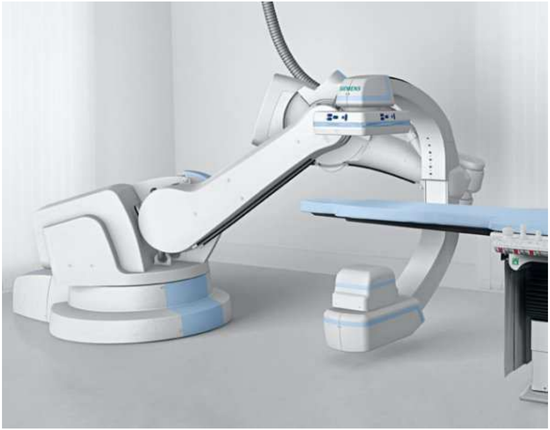



Figure: November 2007: C-arm controlled by a robot arm (image: Siemens Healthcare)

# 3-D Reconstruction in Angiography

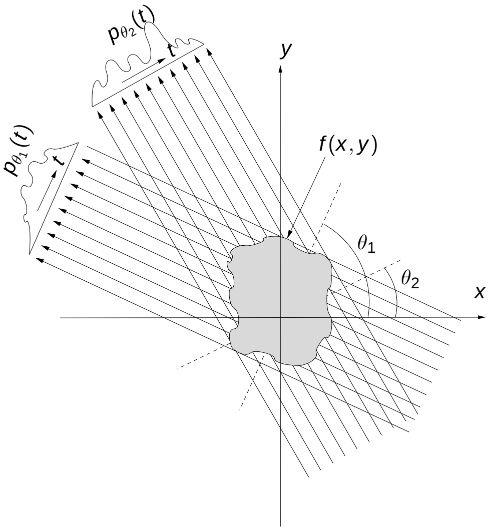
Figure: Standard bi-plane C-arm device (image: Siemens Healthcare)

# Computer Tomography System

Figure: Standard CT system (image: Siemens Healthcare)

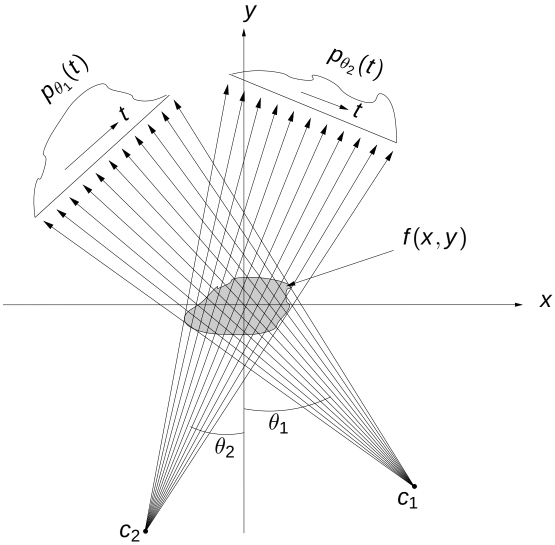
# Diagnostic Medical Image Processing

# Parallel X-Ray Projections




- $p_\theta(t)$: projection under viewing angle $\theta$
- $f(x,y) = f(\boldsymbol{x})$: 2D slice

# Fan Beam Acquisition Geometry



# Cone Beam Acquisition Geometry

In recent years the number of detector lines is increasing. Cone beam acquisition geometry became standard.

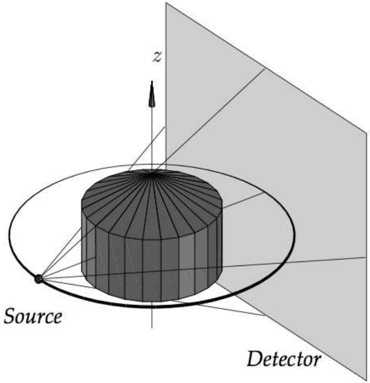



Figure: Circle trajectory: Illustration of cone beam geometry (image: H. Turbell)

# Cone Beam Acquisition Geometry

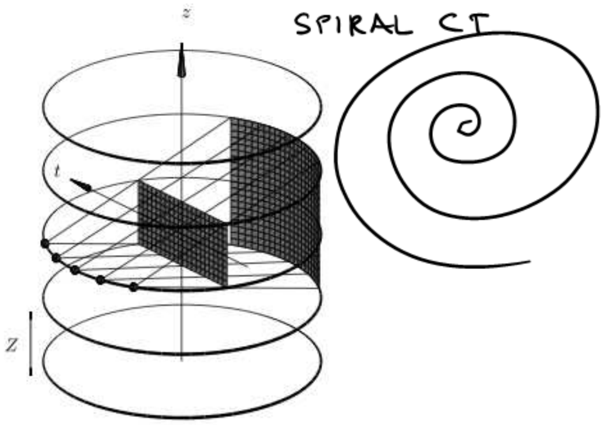



Figure: Helix trajectory: Illustration of cone beam geometry (image: H. Turbell)

# Diagnostic Medical Image Processing

# Sinogram

## Definition

The projections $p_\theta(t)$ can be considered as bivariate functions in $\theta$ and $t$. The resulting image in $(\theta, t)$ is called *sinogram*.

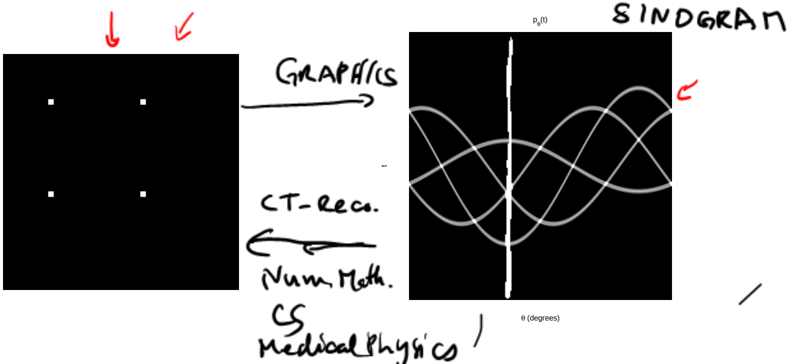



Figure: Intensity image showing four points (left) and the resulting sinogram (right)

# Sinogram

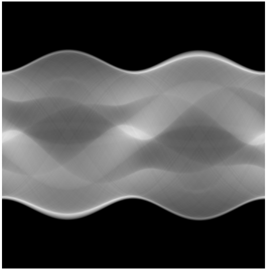


Figure: Shepp Logan Phantom (left) and the resulting sinogram (right)

# Sinogram

... and, of course, you also can do some fun stuff:

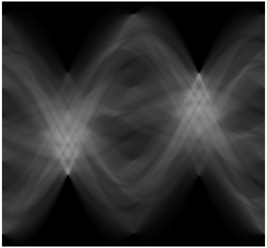


Figure: A portrait of Conrad Wilhelm Röntgen and the computed sinogram

In simple terms we are now looking for an algorithm that allows us to compute the transform from sinogram to the original image.

# Fourier Slice Theorem

$f(\vec{x}) \quad R\vec{x} \longrightarrow f(R\vec{x}), \quad FT(\vec{u})$ for $R_\theta\binom{\xi}{0} \to FT\left(R_\theta\binom{\xi}{0}\right)$

Using the observation about rotated Fourier transforms, we conclude

$$\begin{aligned} FT_1(p_\theta)(\xi) &= FT_2(f \circ \boldsymbol{R}_\theta)(\xi, 0) \\ &= (FT_2(f) \circ \boldsymbol{R}_\theta)(\xi, 0) \\ &= FT_2(f)(\boldsymbol{R}_\theta(\xi, 0)^\mathsf{T}). \end{aligned}$$

## Theorem

***Fourier Slice Theorem****: The Fourier transform of the 1-D projection is equal to the 2-D Fourier transform of the original function along the line through the origin that is parallel to the projection resp. orthogonal to the projection direction.*

# Fourier Slice Theorem

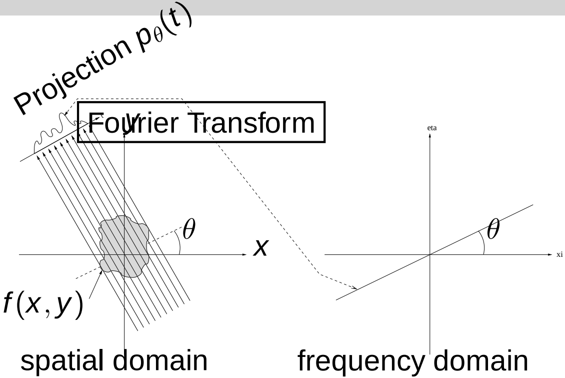



Figure: Illustration of the Fourier Slice Theorem

**Note:** The projections are nearly independent. The only information shared by all projections is the 2-D Fourier transform at (0,0).

# Fourier Slice Theorem

A few comments on the Fourier Slice Theorem:

- It is rather surprising that there is such a remarkably simple relationship between 1-D and 2-D Fourier transform.
- The 1-D Fourier transform of the projections allows for sampling the 2-D Fourier transform of the function to be reconstructed.
- The Fourier Slice Theorem is still the base for mostly all commercially available CT scanners.
- Due to the Fourier Slice Theorem, it is obvious that in the presence of a parallel projection model at least a $\pi$-rotation around the object is required for object reconstruction.
- The straightforward application of the Fourier transform is prohibited due to the fact that it results in a 2-D Fourier transform sampled in polar coordinates.

# Sampling in Frequency Domain

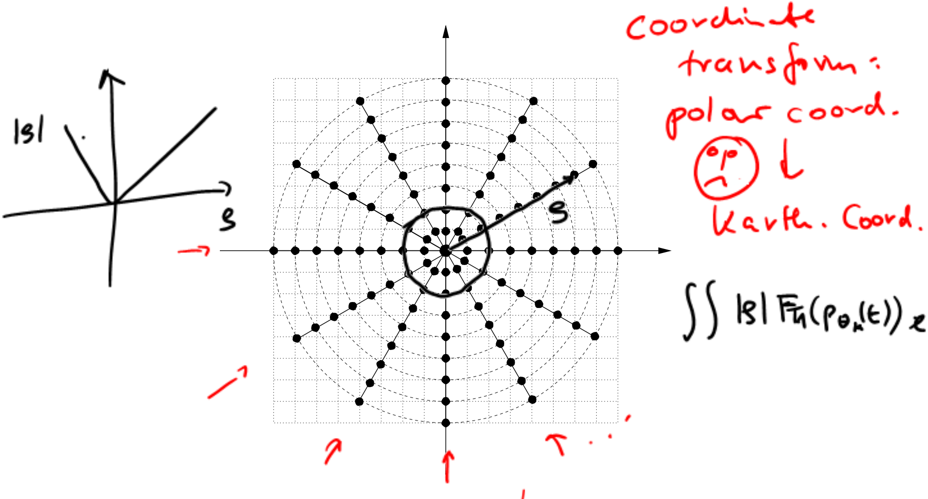



Figure: 2-D Fourier transform is sampled in polar coordinates

# Diagnostic Medical Image Processing

# Filtered Backprojection

Now we have to incorporate the coordinate transform from polar to Cartesian coordinates. We set:

$$\xi \;=\; \varrho \cos\theta \quad \text{and} \quad \eta \;=\; \varrho \sin\theta$$

and get

$$\begin{aligned} f(x,y) &= \int_0^{\pi} \int_{-\infty}^{+\infty} |\varrho|\, F(\varrho\cos\theta, \varrho\sin\theta) \exp\left(a(x,y,\varrho,\theta)\right) d\varrho\, d\theta \\ &= \int_0^{2\pi} \int_0^{+\infty} |\varrho|\, F(\varrho\cos\theta, \varrho\sin\theta) \exp\left(a(x,y,\varrho,\theta)\right) d\varrho\, d\theta \end{aligned}$$

where

$$a(x,y,\varrho,\theta) \;=\; 2\pi i \left(x\varrho\cos\theta + y\varrho\sin\theta\right) \quad .$$

# Filtered Backprojection

Given the acquisition geometry of parallel projections, the reconstruction problem and the Fourier Slice Theorem, we know that:

$$F(\varrho\cos\theta, \varrho\sin\theta) = FT_1(p_\theta)(\varrho)$$

and thus this yields

$$f(x,y) = \int_0^\pi \int_{-\infty}^{+\infty} |\varrho| FT_1(p_\theta)(\varrho) \exp\left(a(x,y,\varrho,\theta)\right) \, d\varrho \, d\theta \quad .$$

The definition of the 1-D Fourier transform now results in:

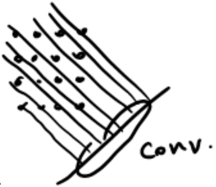
$$f(x,y) = \int_0^\pi \int_{-\infty}^{+\infty} |\varrho| \underbrace{\int_{-\infty}^{+\infty} p_\theta(t) \exp\left(-2\pi i \varrho t\right) \, dt}_{\text{1-D FT}} \cdot \exp\left(a(x,y,\varrho,\theta)\right) d\varrho \, d\theta$$

# Filtered Backprojection

This integral can be rewritten in form of a convolution followed by an integration:

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{+\infty} p_\theta(t) \underbrace{\int_{-\infty}^{+\infty} |\varrho| \exp(2\pi i \ \varrho(x\cos\theta + y\sin\theta - t)) d\varrho}_{k(x\cos\theta + y\sin\theta - t)} dt d\theta$$

$$= \int_0^{\pi} (p_\theta \star k)(x\cos\theta + y\sin\theta)\, d\theta$$




where $\star$ denotes the convolution operator, i.e.

$$(g \star h)(\tau) = \int_t g(t) h(\tau - t)\, dt \quad .$$

# Filtered Backprojection

The function

$$k(x) = \int_{-\infty}^{+\infty} |\varrho| \exp(2\pi i \, \varrho x) d\varrho$$

is called ramp filter due to its shape in Fourier domain.

Using this result, we now have:

**Reconstruction = Convolution of input signal with ramp filter kernel followed by numerical integration.**

The discrete version of the convolution kernel $k$ is important for the final image quality. In practice there are several choices to approximate the kernel. The kernel basically depends on the tissue class to be reconstructed.

# Different Kernels for Convolution

Different discrete convolution kernels ($t = 1 \dots N_\mathrm{t}$) are:

- no filtering (nice try, but never use it):

$$k[t] \quad := \quad \mathrm{id} \tag{4}$$

- high pass filtering using backward differences (better, but no option):

$$k[t] \quad := \quad p[t] - p[t-1] \tag{5}$$

- Shepp–Logan filtering:

$$k[t] \quad := \quad \frac{-2}{\pi(4t^2 - 1)} \tag{6}$$

# Different Kernels for Convolution

- RamLak filtering:

$$g[t] := \begin{cases} \frac{\pi}{4}, & \text{if} \quad t = 0 \\ 0, & \text{if} \quad t \text{ even} \\ \frac{-1}{\pi t^2}, & \text{otherwise} \end{cases} \tag{7}$$

# Discrete Sampling

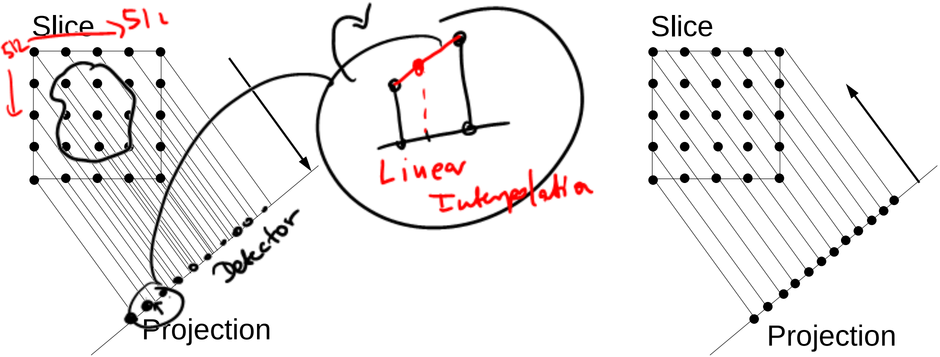



Figure: Different sampling methods for a 2-D slice to be reconstructed: sample the result (left) vs. sample the input projection (right) - here for orthographic projection

**Rule of thumb:** Always sample in the space where you expect the result.

# Discrete Version of Filtered Backprojection

- uniform sampling of the interval of $\theta$:

$$\Delta\theta \;=\; \frac{2\pi}{N_\mathrm{v}} \tag{8}$$


- the $i$–th angle is defined by:

$$\theta_i \;=\; i \cdot \Delta\theta \quad \text{where} \quad i = 1, \ldots, N_\mathrm{v} \tag{9}$$

- let $N_t$ be the number of detector elements and $\tau$ the length of the detector


- uniform sampling of the detector results into:

$$\Delta t = \frac{\tau}{N_t} \tag{10}$$

- number of slice columns is $N_\mathrm{x}$ and number of slice rows is $N_\mathrm{y}$

# Filtered Backprojection

| | | |
|---|---|---|
| Initialize slice: $f[y][x] := 0$ for all $x, y$ | | |
| FOR $i := 1 \ldots N_{\mathrm{v}}$ | | // $N_{\mathrm{v}}$ number of angles |
| $\theta_i := i \cdot \Delta_\theta$ | | |
| $p[t] = p_{\theta_i}(t)$ | | |
| convolution: $h = p \star g$ | *filtering* | // $g$ e.g. from (**??**) |
| | *backprojection (numerical integration)* | |
| FOR $y := 0 \ldots N_{\mathrm{y}} - 1$ | | // iterate over all slice rows |
| FOR $x := 0 \ldots N_{\mathrm{x}} - 1$ | | // iterate over all slice columns |
| compute projection of $(x, y)$ in the observed 1–D signal: $t_{x,y}$ | | |
| compute $h[t_{x,y}]$ (by interpolation) | | |
| increment $f[y][x]$ by $h[t_{x,y}]$ | | |
| Output $s$ | | |

Figure: Filtered backprojection module for one slice

# Filtered Backprojection Examples

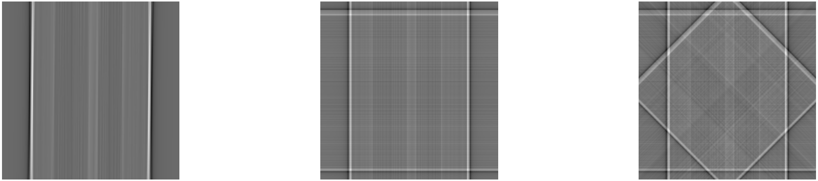


Figure: Filtered backprojection with Shepp-Logan filter (angle increment: 180 degrees (left), 90 degrees (middle), 45 degrees (right))

# Filtered Backprojection Examples

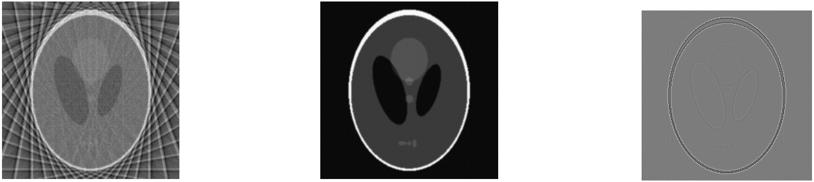


Figure: Filtered backprojection with Shepp-Logan filter (angle increment: 10 degrees (left), 1 degree (middle), difference image of original Shepp-Logan phantom and reconstruction result with 1 degree increment (right))

# Filtered Backprojection Examples

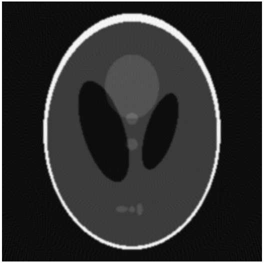
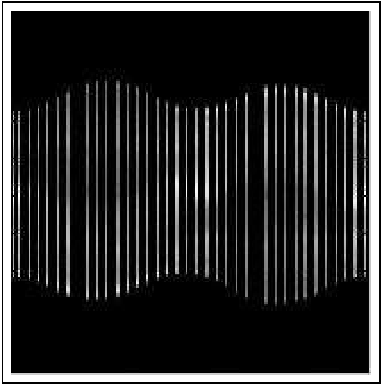


Figure: RamLak filtering (left) and no filtering (right) followed by backprojection

# Missing Projections

**Problems**:

- Can we reconstruct if some projections are missing?
- To which extend is interpolation working?



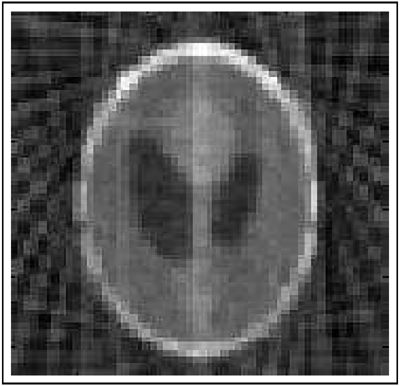


Figure: Incomplete projection data: sinogram (left), FBP reconstruction result (right) with obvious artifacts (stripes)

# Missing Projections

**Idea: Interpolation of sinogram values**

- missing projections lead to missing columns in the sinogram
- observed sinogram is *complete sinogram* multiplied with *defect sinogram* (0/1–values)
- application of **defect interpolation** to sinogram

# Missing Projections

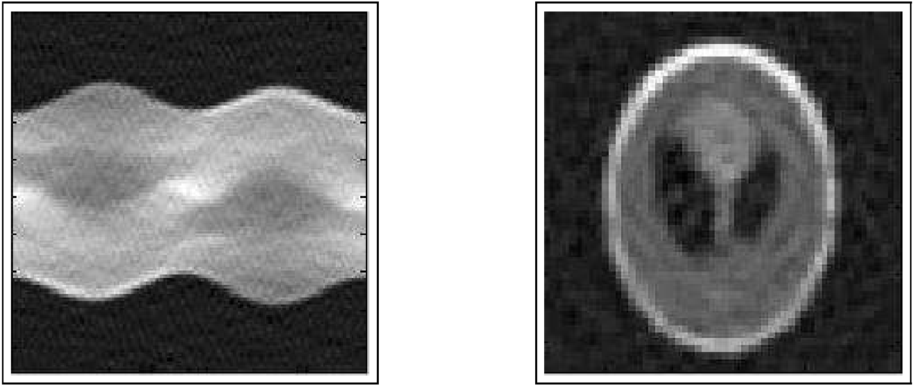


Figure: Interpolated sinogram (left) and reconstruction result (right)

# Take Home Messages

Beer's Law

- X-ray attenuation law
- Innovation drivers in CT
- Fourier Slice Theorem
- Empirical result: 5 out of 10 students do **not** know the Fourier Slice Theorem in oral exam.
- Filtered backprojection
- Computational complexity of filtered backprojection

# Further Readings

- For practitioners we recommend the book of A.C. Kak and M. Slanely;
  A.C. Kak and Malcolm Slaney: *Principles of Computerized Tomographic Imaging,* Society of Industrial and Applied Mathematics, 2001 (download here)
- A nice overview of CT can be found in the PhD thesis: Henrik Turbell: Cone-Beam Reconstruction Using Filtered Backprojection, Linkoping University, Sweden, February 2001.(download here)

# Further Readings

- The exact reconstruction method is introduced in Alexander Katsevich: *Theoretically exact filtered backprojection-type algorithm for spiral computed tomography*, SIAM Journal of Applied Mathematics, 62:2012-1026, 2002.
- Radon's original paper where he published his famous inversion formula:
  J. Radon: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, Bericht der Sächsischen Akademie der Wissenschaft, Leipzig Math. Phys. Kl., 69(1917), pp. 262-267.