Frequency-space image processing

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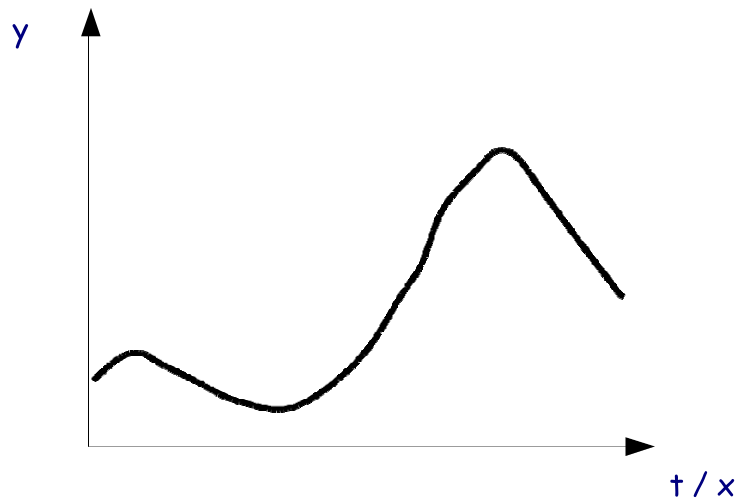
The analog signal



- signal is a record of a certain variation value y depending on the elapsed time

$$y(t)=f(t)$$

- the function f can be given analytically (equation) or not
- independent variable t (time) can also mean the distance, the position x
- such a signal can impose a number of conditions (continuity, differentiability, ...)

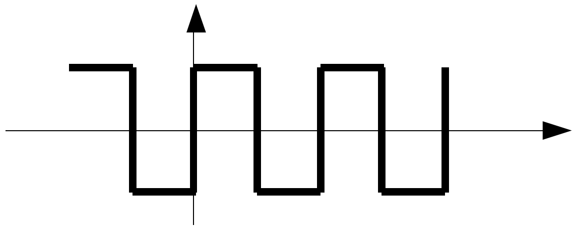
Signal distribution for basis functions

- each signal can be replaced by a linear combination of other functions called **basis functions**:

$$y(t)=a_1\, f_1(t)+a_2\, f_2(t)+\ldots+a_n\, f_n(t) \quad (+\ldots)$$

- sometimes it is a finite sum, sometimes not
- asuume the signal given by the formula:

$$y(t)=\begin{cases}1 & \text{if} \quad k\cdot 2\pi \leqslant t < k\cdot 2\pi+\pi \\ 0 & \text{if} \quad k\cdot 2\pi+\pi < t \leqslant (k+1)\cdot(2\pi)+\pi\end{cases}$$

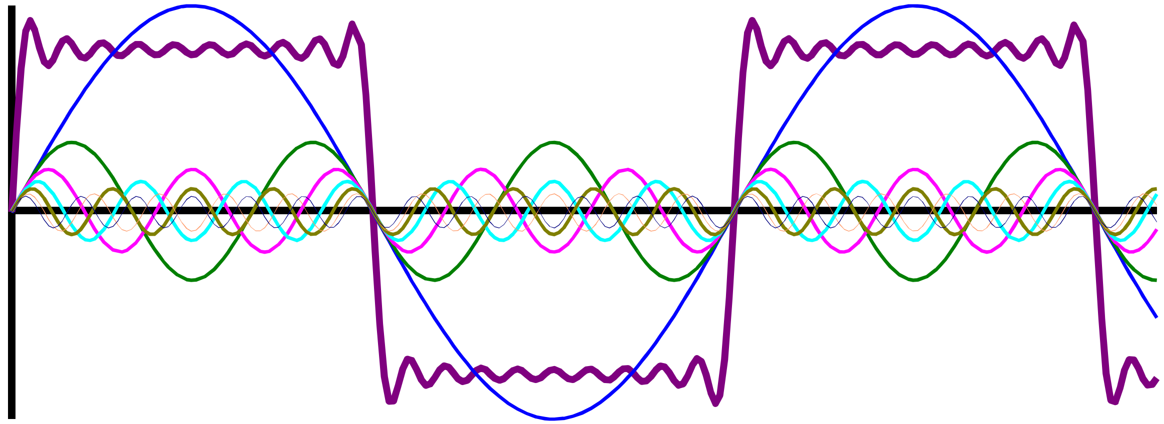


- such a signal can be represented by an infinite sum of the sines of increasing frequency:

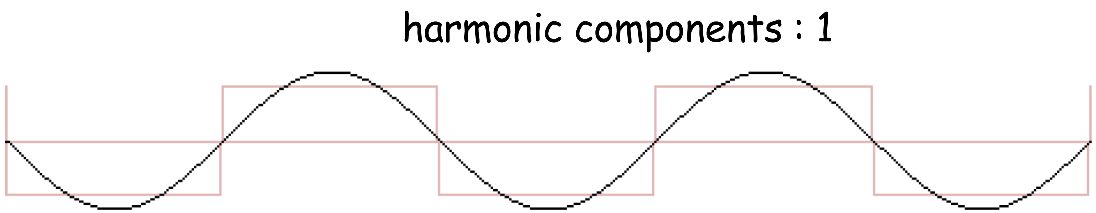
$$\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\frac{1}{7}\sin(7x)+\ldots=\sum_{i=0,\ldots,\infty}\frac{1}{2i+1}\sin((2i+1)x)$$

- if we take a finite number of terms, it will be only an approximation of the signal (the more accurate the more ingredients)

Signal distribution for basis functions



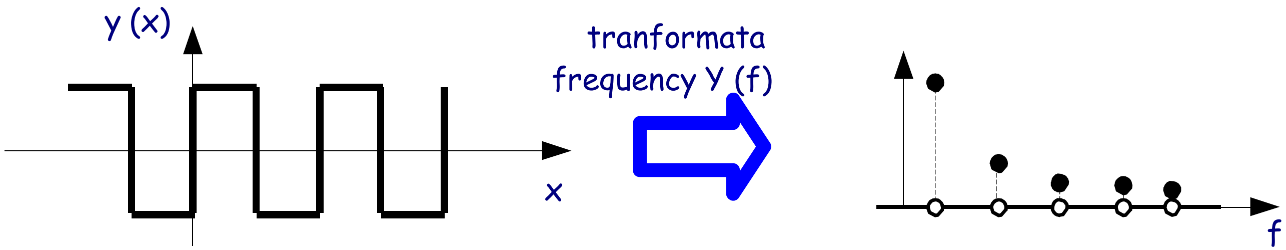
- on the graph is the sum of 10 components (i = 0, ..., 9) (not all components of the sines are drawn)
- animation showing the impact of the components on the quality of approximation



Fullofstars, Wikipedia.org

Frequency transform

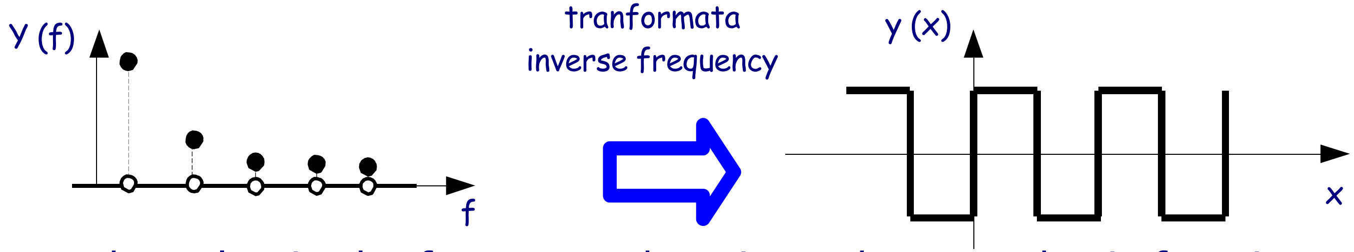
- If we assume that the base functions are in this case a sine function with increasing frequency (**harmonic functions**), it can be seen that we have made the transition from the **time domain** to **the frequency domain**
- the function values in the frequency domain are all zeros except for the frequency f = 1 (value 1), f = 3 (value 1/3), f = 5 (value 1/5), ...



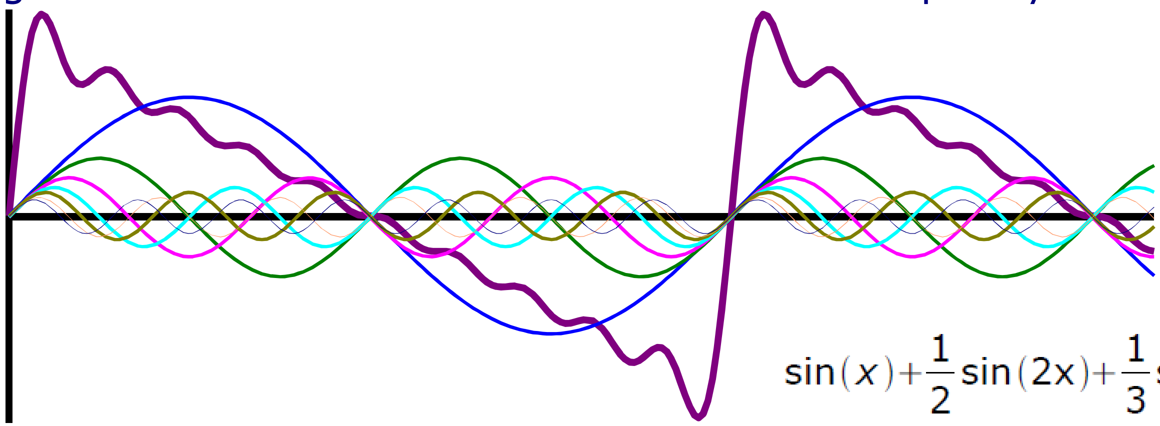
- the use of other basis functions give other values
- this transformation (transform) is reversible - again, we can go from the frequency domain to the time domain (spatial)

The inverse transform of the frequency

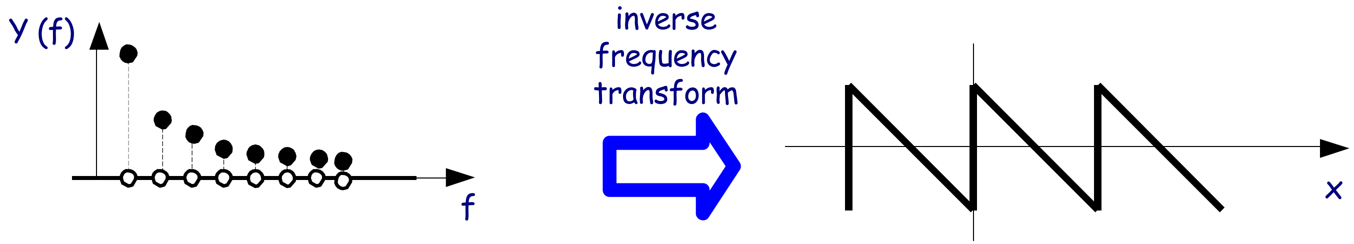
- This transformation is reversible



- change the value in the frequency domain at the same basis function may give a different form of the function in the frequency domain:

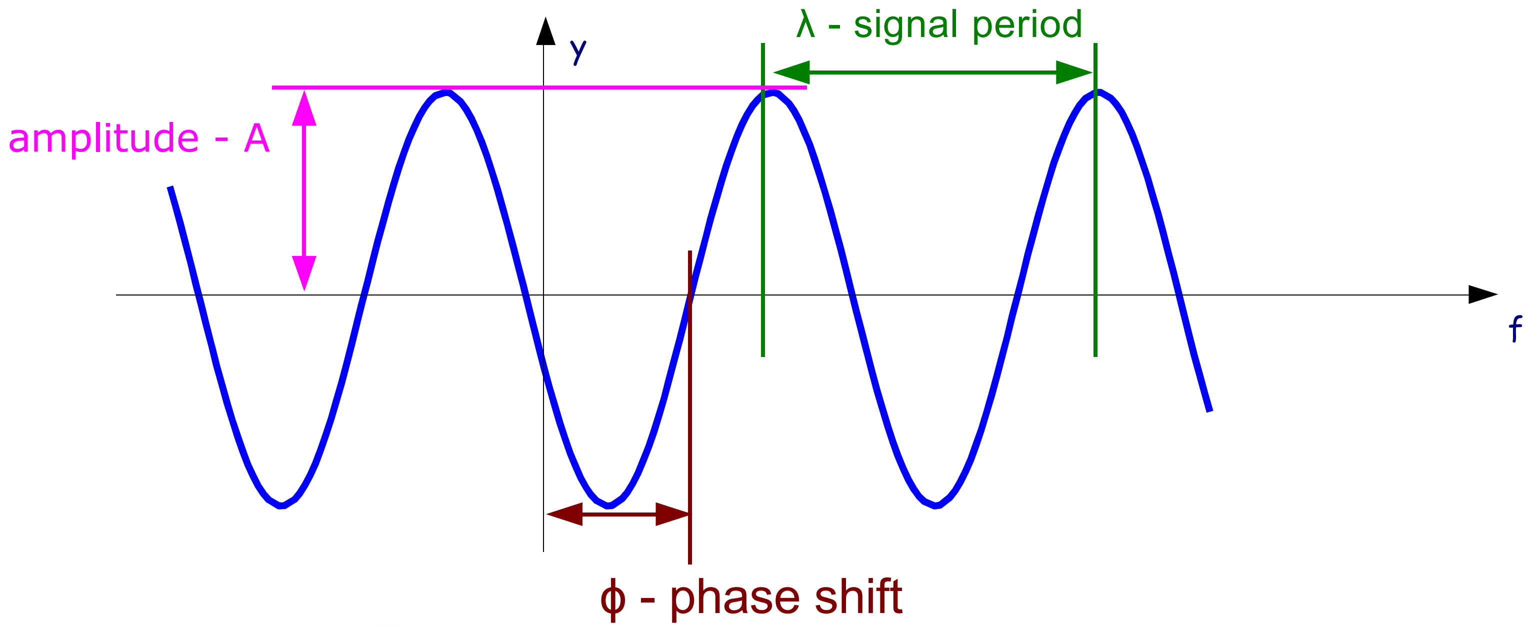


$$\sin(x)+\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)+\frac{1}{4}\sin(4x)+\ldots$$



Characteristics of the sine wave

$$y(x) = A\sin\left(\frac{2\pi}{\lambda}x - \phi\right)$$



$\omega = \frac{2\pi}{\lambda}$ - the angular frequency [radians]

$f = \frac{1}{\lambda}$ - the frequency [Hz]

The Fourier transform

- Fourier transform is an example of the frequency transform
- basis functions here are functions of sines and cosines of increasing frequency (harmonic functions)

$$y(x)=A_0+\sum_{k=1}^{\infty} A_k \cos(\frac{2\pi k}{\lambda}x)+B_k \sin(\frac{2\pi k}{\lambda}x)$$

where: λ - signal period

- coefficients A and B are **complex**!
- complex factor encodes the amplitude and phase of the basis functions

$$y(x)=A_0+\sum_{k=1}^{\infty} |A_k| \cos(\frac{2\pi k}{\lambda}x+\phi_{A_k})+|B_k| \sin(\frac{2\pi k}{\lambda}x+\phi_{B_k})$$

$$|A_k|=\sqrt{\mathrm{re}(A_k)^2+\mathrm{im}(A_k)},\ \phi_{A_k}=tg^{-1}\frac{\mathrm{im}(A_k)}{\mathrm{re}(A_k)},\quad B_k \text{ and } \phi_{B_k}$$

- equivalent form:

$$y(x)=\sum_{k=-\infty}^{\infty} C_k e^{i\frac{2\pi k}{\lambda}x}=\sum_{k=-\infty}^{\infty} |C_k| \{\cos(\frac{2\pi x}{\lambda}x+\phi_k)+i\ \sin(\frac{2\pi x}{\lambda}x+\phi_k)\}$$

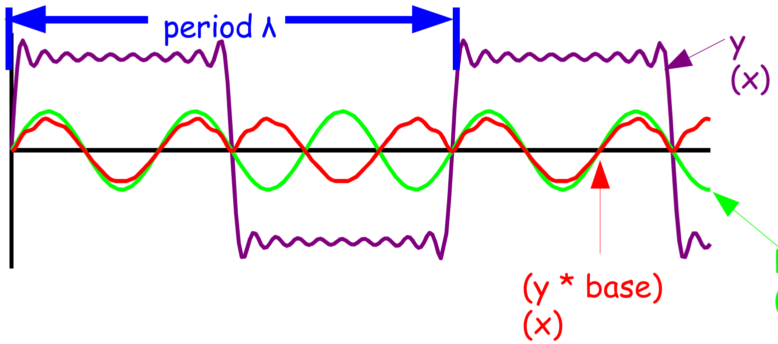
where the coefficients C are complex

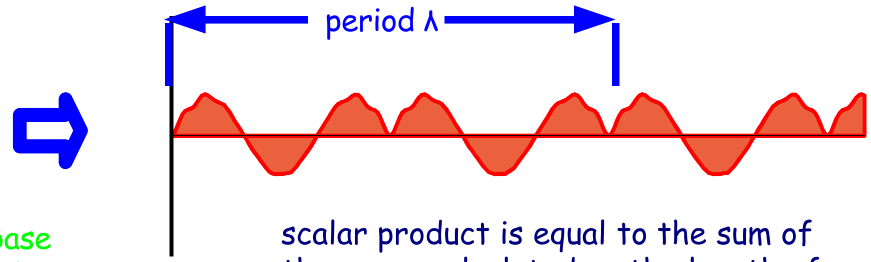
$$e^{\pm i\cdot x}=\cos(x)\pm i\ \sin(x)$$

Calculating the coefficients

$$C_k = |C_k| e^{+i\phi_k} = \frac{1}{\lambda}\int_{-\lambda}^{\lambda} y(x) e^{-i\frac{2\pi k}{\lambda}x} dx$$

- the coefficients C measure the similarity function y (x) for each of the basis functions
- similarity measure is calculated by the integral of the scalar product of two functions





- the function y (x) is considered to be periodic

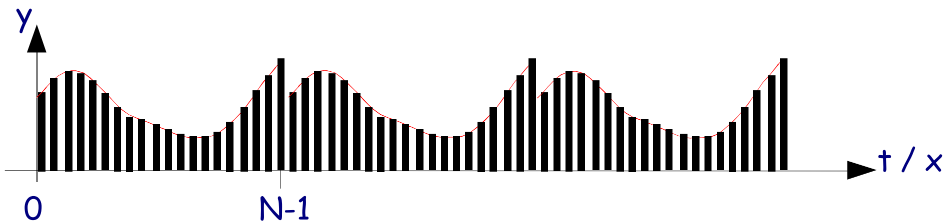
$$y(x) = y(x + k \cdot \lambda)$$

- Fourier transform is reversible:

$$Y(f) = \int_{-\infty}^{\infty} y(x) e^{-i2\pi f x} dx \qquad y(x) = \int_{-\infty}^{\infty} Y(f) e^{i2\pi f x} df$$

Discrete Fourier Transform

- in the case of discrete (digital signal, the digital image) integrals turn to sums
- basis functions also have the discrete form
- digital signal is treated as one period and supplemented with front and rear their repetition - it can introduce discontinuities

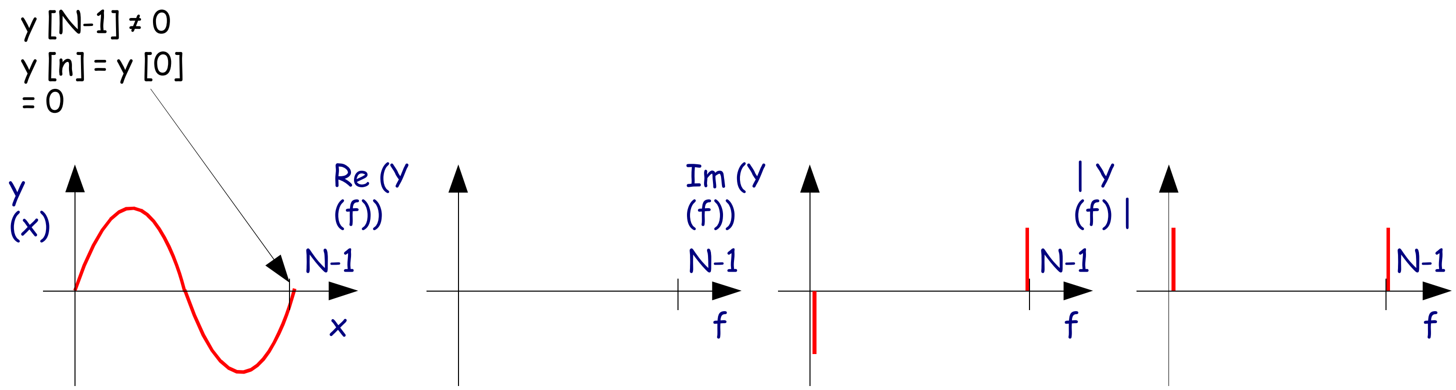


- number of samples is determined to N - it is also a function of time

$$y(x)=\sum_{k=0}^{N-1} C_k e^{i\frac{2\pi k}{N}x} \qquad Y(f)=\sum_{k=0}^{N-1} C_k e^{-i\frac{2\pi k}{N}f}$$

$$C_k=\frac{1}{N}\sum_{x=0}^{N-1} y(x) e^{-i\frac{2\pi k}{N}x}$$

The Fourier transform of the sine



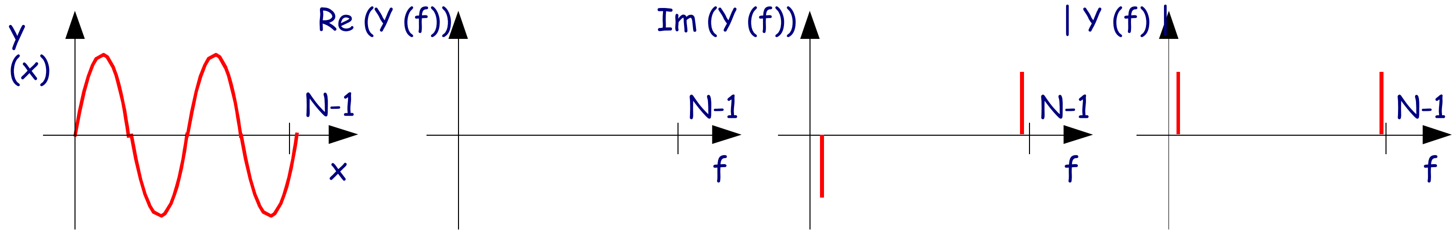
$$y[x]=\sin(\frac{2\pi}{N}x)$$

$$Y[0]=0$$
$$Y[1]=0-C\cdot i$$
$$Y[N-1]=0+C\cdot i$$

$$\sqrt{\Re(Y(t))^2+\Im(Y(t))^2}$$

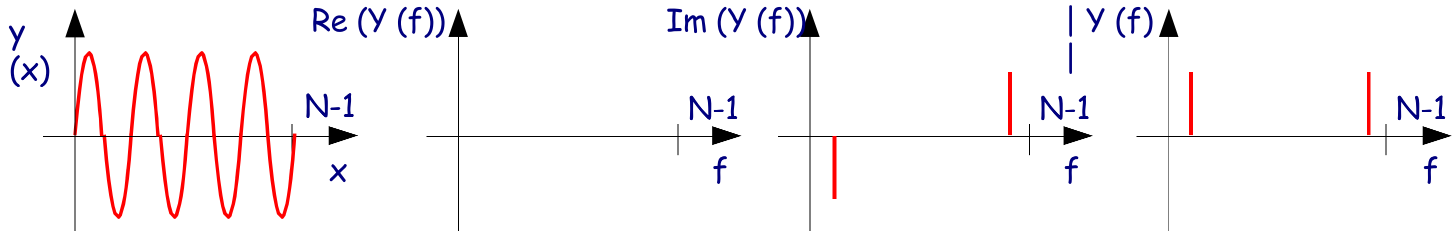
- sinus has a frequency f = 1 (one run at the N samples)
- Creates two "peaks" in the DFT: of the second component (Y [1]) and the last (Y [N-1]), they are symmetrical in the imaginary part of point (N / 2, 0)

The Fourier transform of the sine



$$y[x]=\sin(\frac{2\pi}{N}2\cdot x)$$

$$Y[2]=0-C\cdot i$$
$$Y[N-2]=0+C\cdot i$$

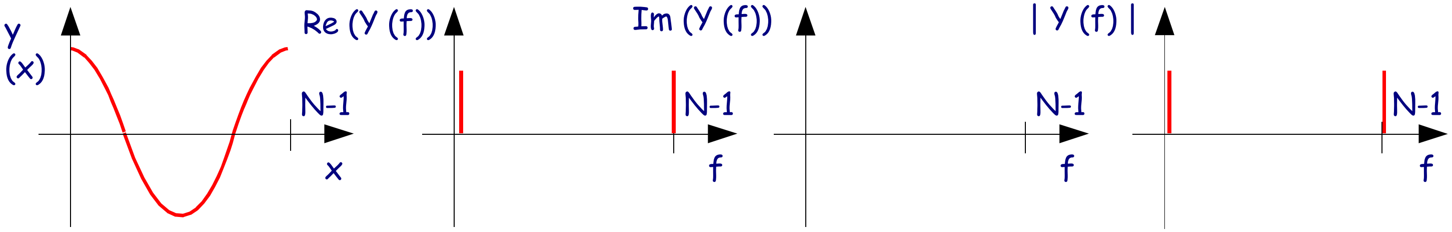


$$y[x]=\sin(\frac{2\pi}{N}2\cdot x)$$

$$Y[4]=0-C\cdot i$$
$$Y[N-4]=0+C\cdot i$$

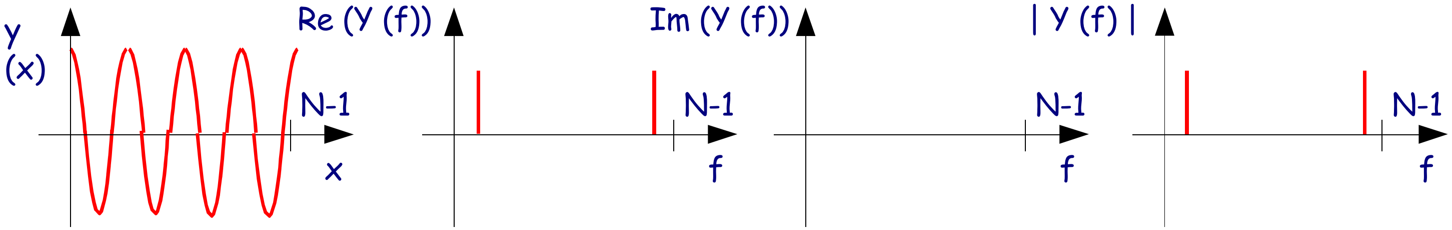
- increasing the frequency moves peaks in the imaginary part in the direction of point (N / 2.0)
- transform module also moves

Cosine Fourier transform



$$y[x]=\cos\left(\frac{2\pi}{N}x\right)$$

$$Y[1]=C+0\cdot i$$
$$Y[N-1]=C+0\cdot i$$



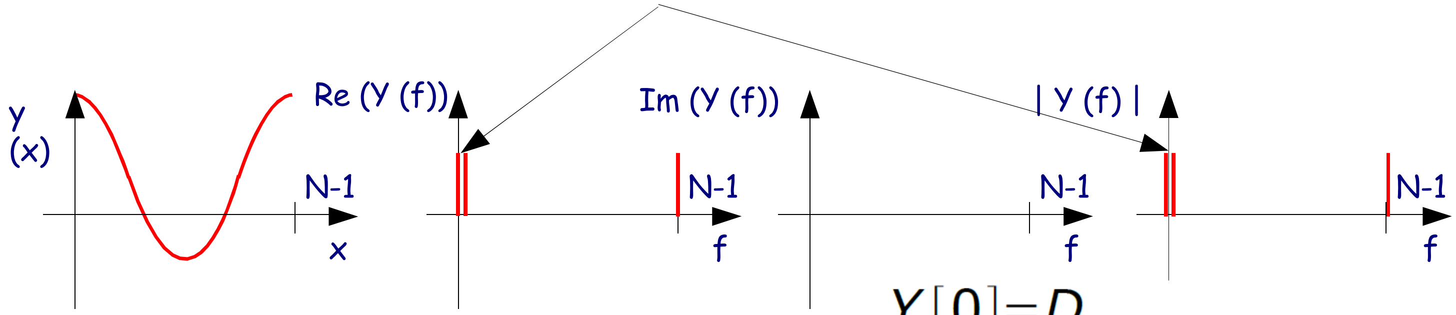
$$y[x]=\cos\left(\frac{2\pi}{N}4\cdot x\right)$$

$$Y[4]=C+0\cdot i$$
$$Y[N-4]=C+0\cdot i$$

- also the position of "peak" represents the frequency
- peaks in the real part are symmetrical with respect to a perpendicular line to the x-axis and passing through the point (N / 2, 0)
- transform amplitude always remains the same

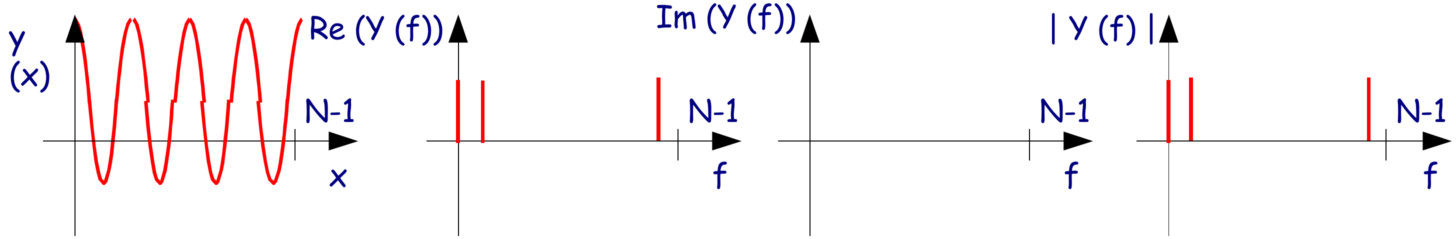
Component of the constant signal

- Y [0] = 0 ⇒ integral under the graph is equal to 0
- it is the frequency 0
- the shifting of function to the top causes the appearance of a real constant component



$$y[x]=\cos(\frac{2\pi}{N}x)+0.5$$

$$Y[0]=D$$
$$Y[1]=C+0\cdot i$$
$$Y[N-1]=C+0\cdot i$$

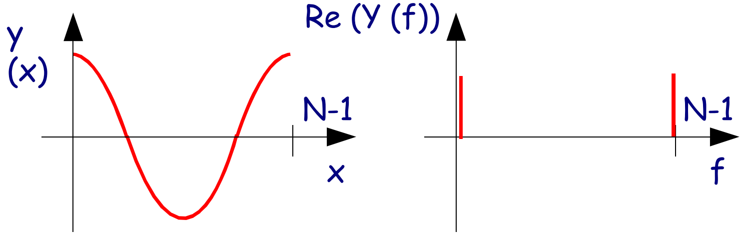


$$y[x]=\cos(\frac{2\pi}{N}4\cdot x)+0,5$$

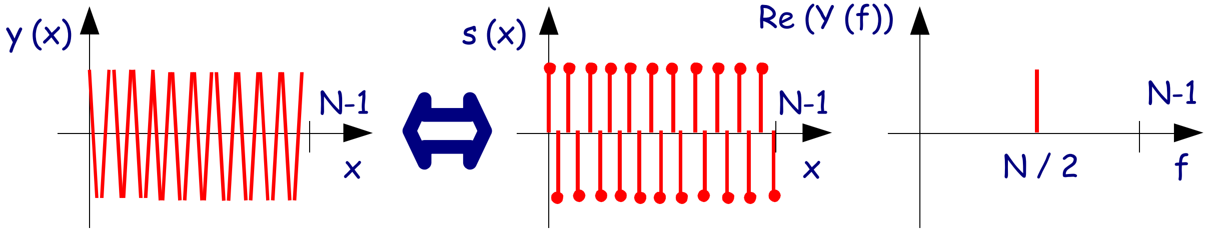
$$Y[0]=D$$
$$Y[4]=C+0\cdot i$$
$$Y[N-4]=C+0\cdot i$$

The frequency range

- the smallest non-zero frequency bearing in mind f_{min} = 1 - one repetition of signal in the window / image

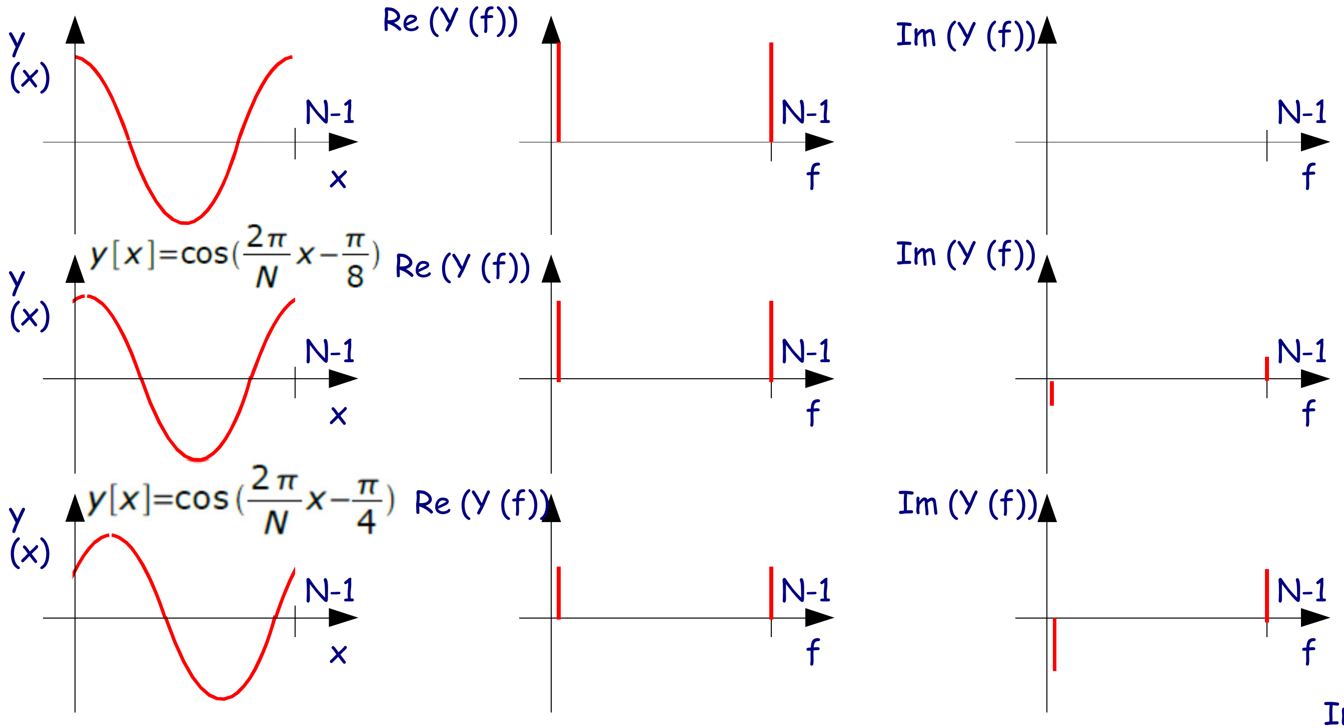


- highest frequency, according to the theory of Nyquista-Shannon sampling is f = N / 2

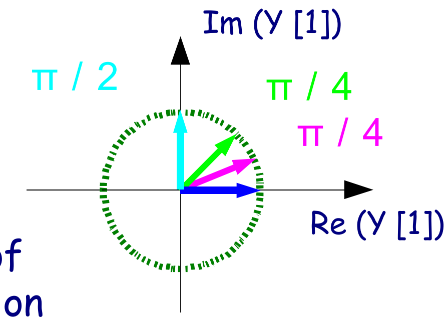


- the frequency f gives in the transform two "peaks": Y [f] and Y [N-F]

Shift in the phase of the signal



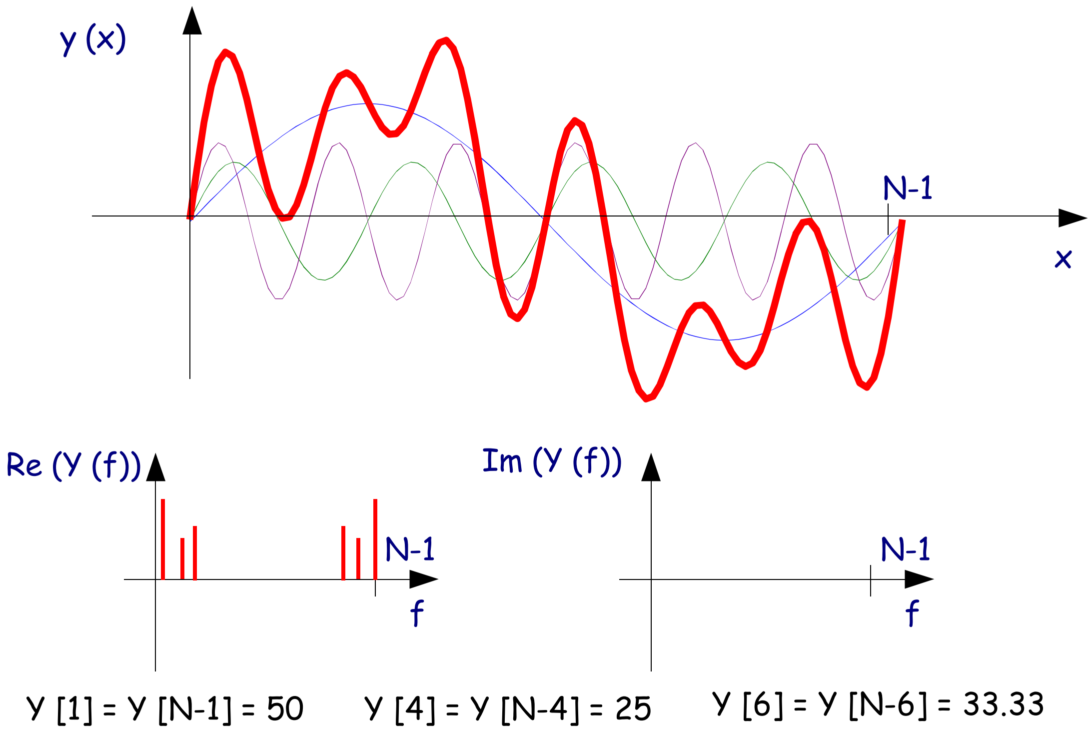
- shifting the phase of π / 2 will give sinus
- when moving the real part decreases and imaginary parts increases
- module stays the same all the time!



influence of phase shift on the value Y [0]

Complex signal

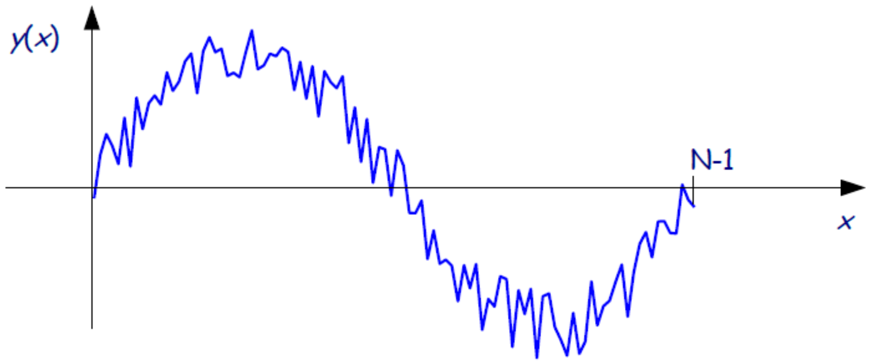
$$y[x]=\cos\left(\frac{2\pi}{N}x\right)+\frac{1}{2}\cos\left(\frac{2\pi}{N}4\cdot x\right)+\frac{2}{3}\cos\left(\frac{2\pi}{N}6\cdot x\right);\quad x=0,\ldots,N-1$$



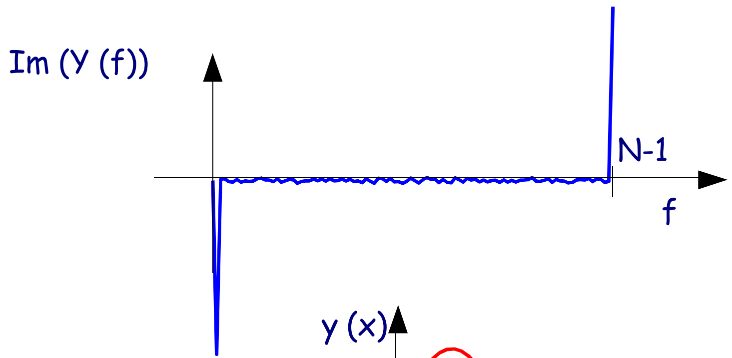
- Complex signal easy separates the frequency domain

Separating signal from noise

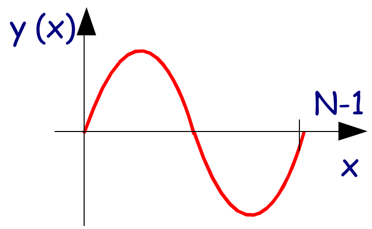
- sine signal with noise



- after Fourier transform:

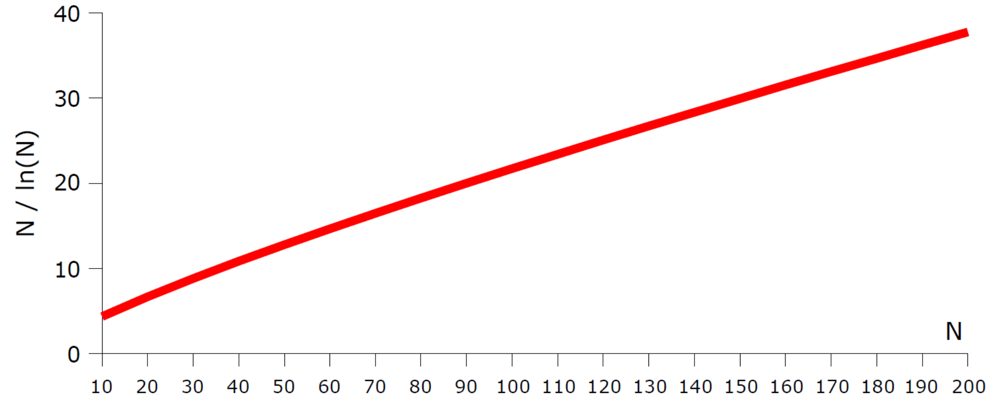


- the thresholded and reverse transform recover the signal without noise:
- noise signal is different frequencies of the lower frequency



The computational complexity of the DFT Fast Fourier Transform (FFT)

- we have to count ~N ratios
- each factor is the sum of ~N expressions
- computational complexity of **DFT**: **O(N²)**
- the complexity of the inverse transform is the same
- Fast Fourier Transform - **FFT** - calculate coefficients of the complexity of **O(N ln N)**!
- for small N it does not really matter
- for large N the differences are very great!



- FFT is rather complicated
- usually are library functions which count the FFT
- original FFT counts for $N = 2^k$ (powers of 2)
- Library procedures to can deal with it

Discrete Fourier Transform 2D

- assume the image size NxN

$$F(a,b)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} I(x,y)\,e^{-i2\pi\left(\frac{a\cdot x}{N}+\frac{b\cdot y}{N}\right)}$$

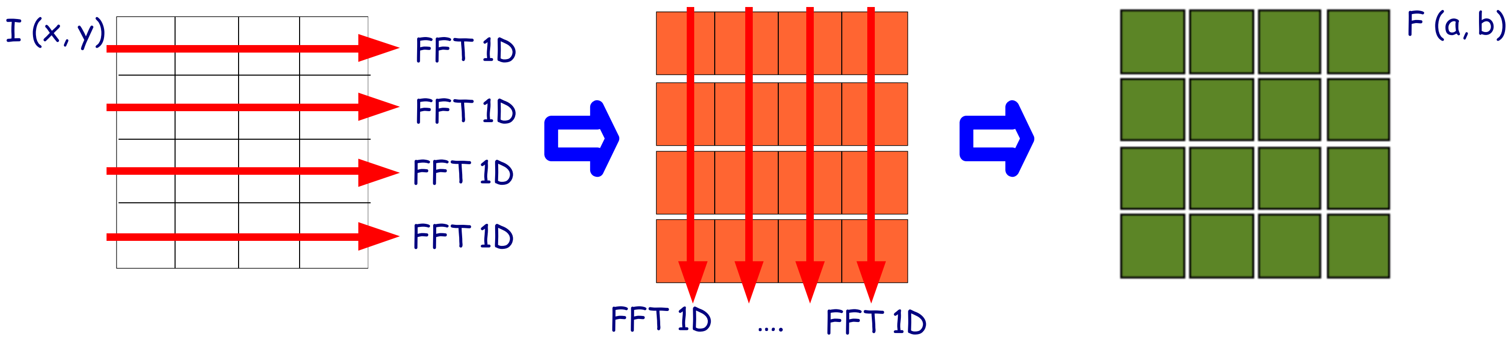
- IDFT 2D (Inverse Fourier Transform 2D):

$$I(x,y)=\frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} I(x,y)\,e^{+i2\pi\left(\frac{a\cdot x}{N}+\frac{b\cdot y}{N}\right)}$$

- Fourier transform is separable with respect to dimensions:

$$F(a,b)=\frac{1}{N}\sum_{y=0}^{N-1} X(a,y)\,e^{-i2\pi\frac{b\cdot y}{N}} \quad \text{where } X(a,y)=\frac{1}{N}\sum_{x=0}^{N-1} I(x,y)\,e^{-i2\pi\frac{a\cdot x}{N}}$$

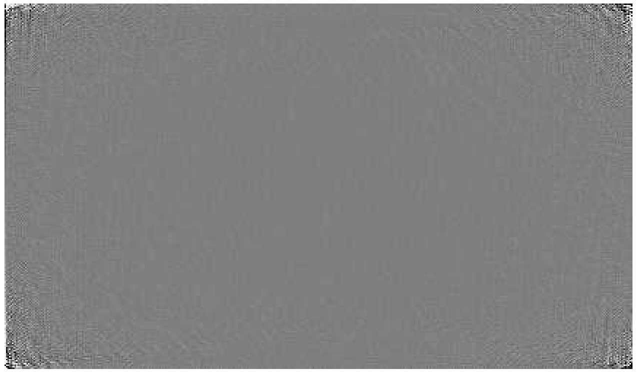
- ie. first can be counted N transformation 1D in N rows and next N transformation 1D in columns (with the results of the first N transform), which accelerates the calculation - O (N^2) rather than O (N^4)
- further acceleration calculations: FFT - O (N ln N)



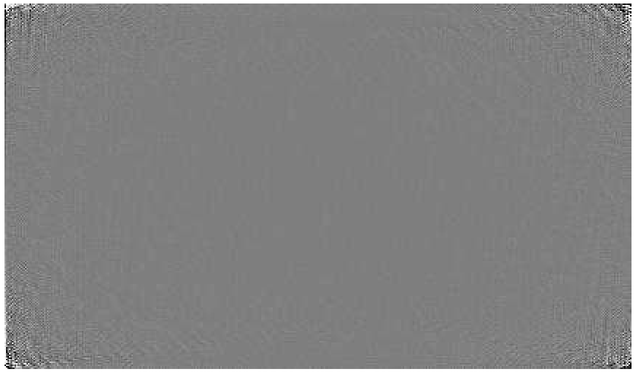
Transform image

Image

Re (FFT (Image))



The (FFT (Image))

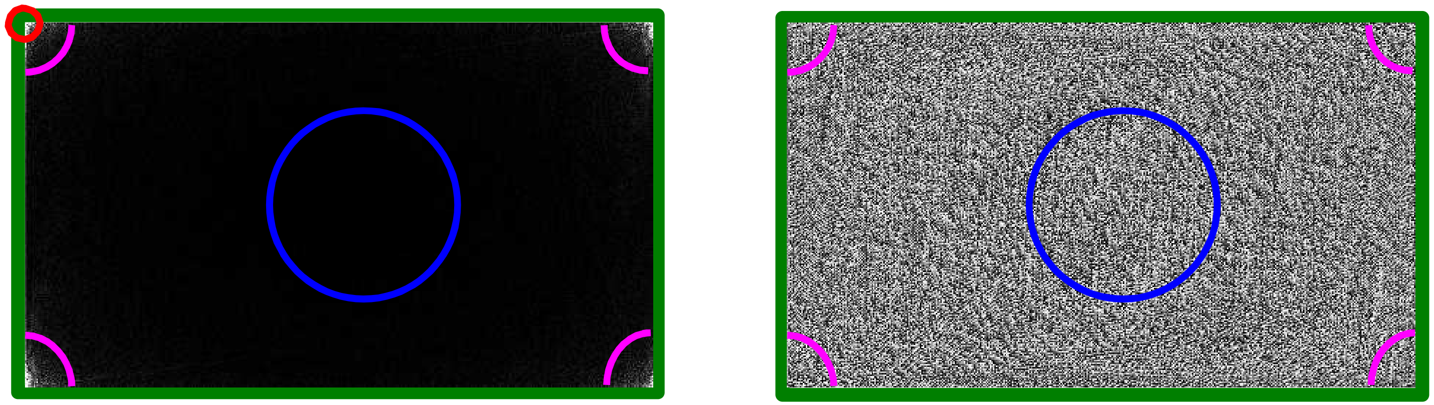


- white - positive, black - negative, gray - zero values
- little can be seen here

Module transform phase

$$|FFT(Image)| = \sqrt{\Re(FFT(Image))^2 + \Im(FFT(Image))^2}$$

$$phase\ FFT(Image) = tg^{-1} \frac{\Im(FFT(Image))}{\Re(FFT(Image))}$$

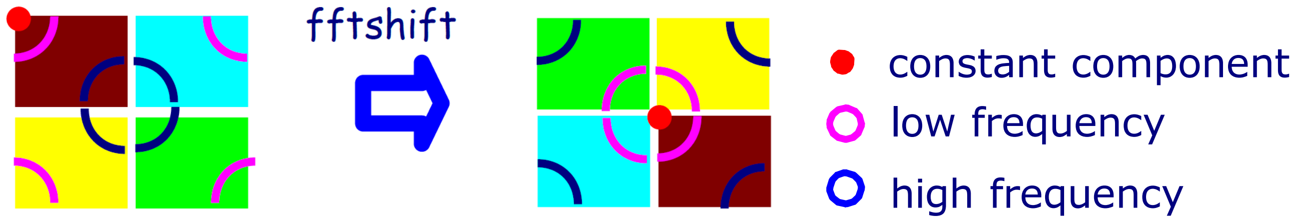


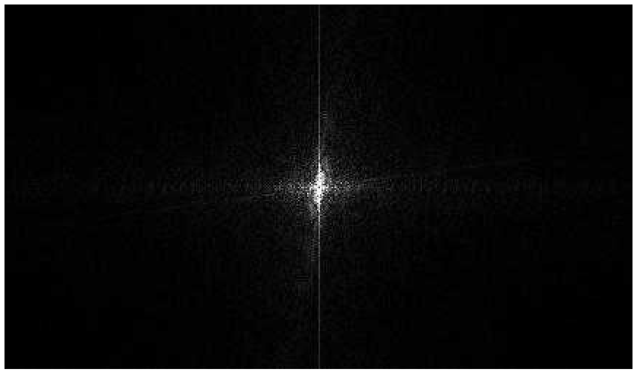
constant component   low frequencies   high frequency

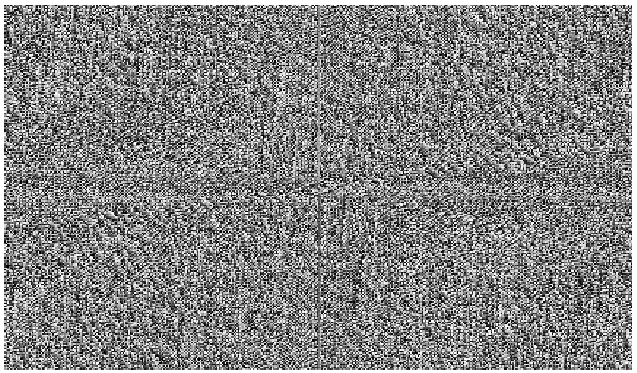
- FFT (image) [0] = 0 and 34721.451 - constant component is large integer (module = 34721.451), phase = 0°
- constant component was much higher than other frequencies (max = 11221)
- low frequencies are more strongly represented than high
- image does not have any noticeable regularity
- most information which is important in the interpretation carries with it the module - it is the most often used

Image shift transform

- the previous figure low frequency (carrying the most useful information) are scattered corners of the transform
- it usually accumulates in the center of the image by shifting quadrants of the image transform - concerns to the real / complex and module / Phase





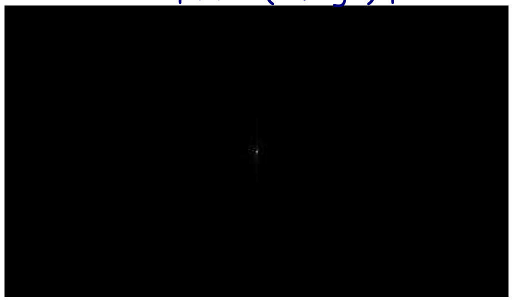


- image module is ordered, the phase image it does not visually changes
- <u>before the inverse transform must return to its initial state</u>

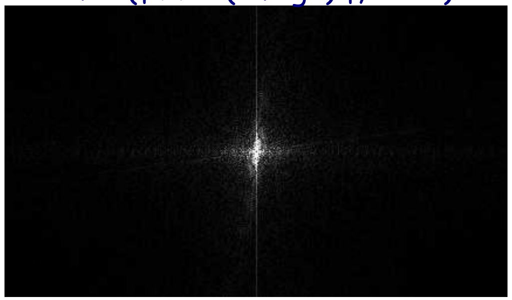
Logarithmic scale module

- component has a much greater value than the other frequencies (in the middle image has been cropped from 34721 to 1000 to other frequencies were not black - as in the left image)
- because in most visualization shows the logarithm of the module (picture right)

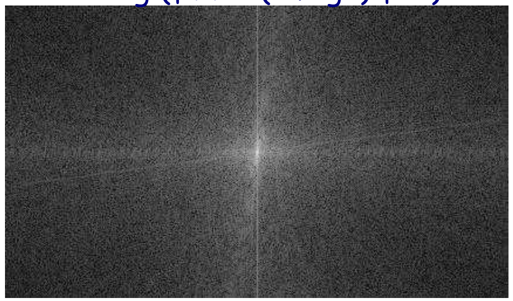
| FFT (Image) |



min (| FFT (Image) |, 1000)



log (| FFT (Image) | +1)



- in the phase image does not matter - the range is (-π , π)
- operation only for visualization

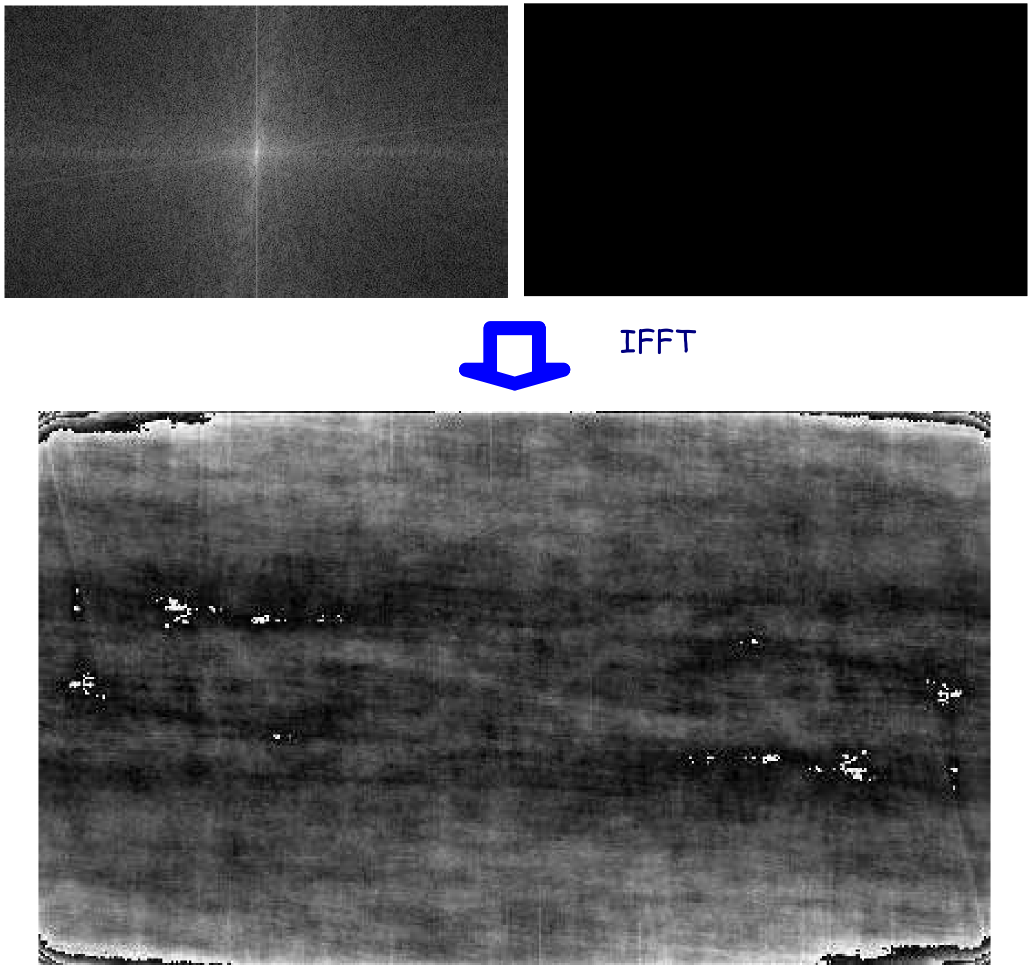
Inverse transform

Image

IFFT (FFT (Image))

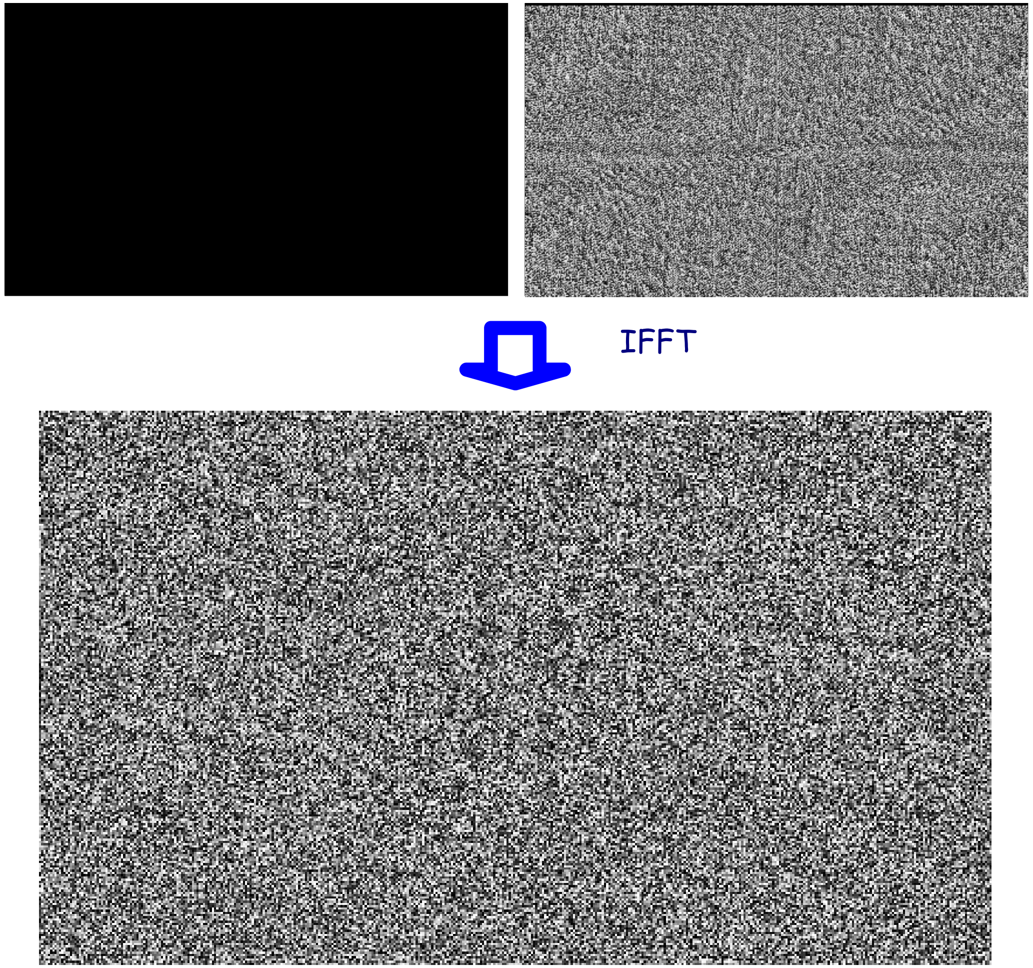
- Fourier transform is iversible: X = IFT (FT (X))
- DFT is only an approximation of the Fourier transform
- hence the DFT appears loss of quality (the effect of approximations)

Inverse transform without phase



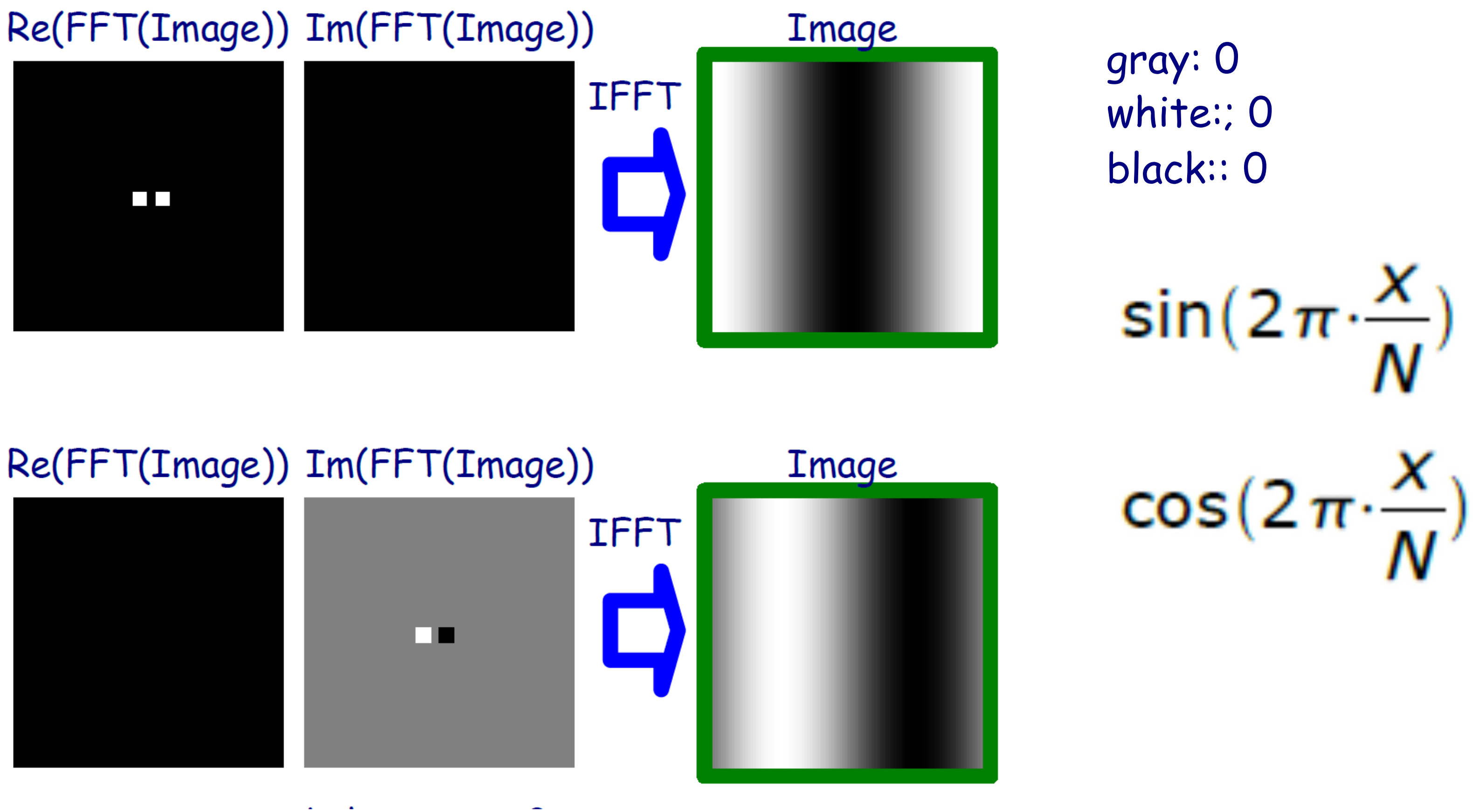
- despite the seeming chaos phase is necessary to reverse transform

Inverse transform without module



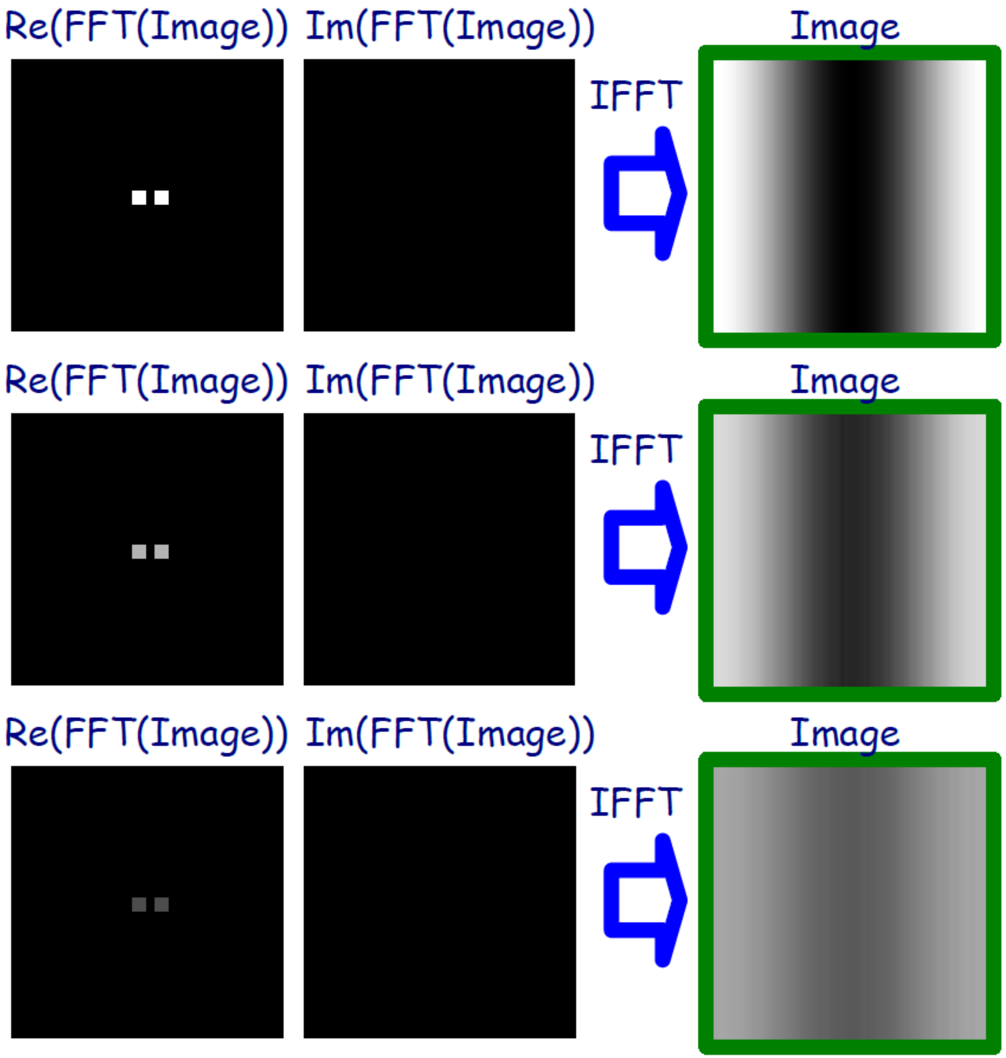
- reconstruction without the module is also impossible

The basis functions of 2D



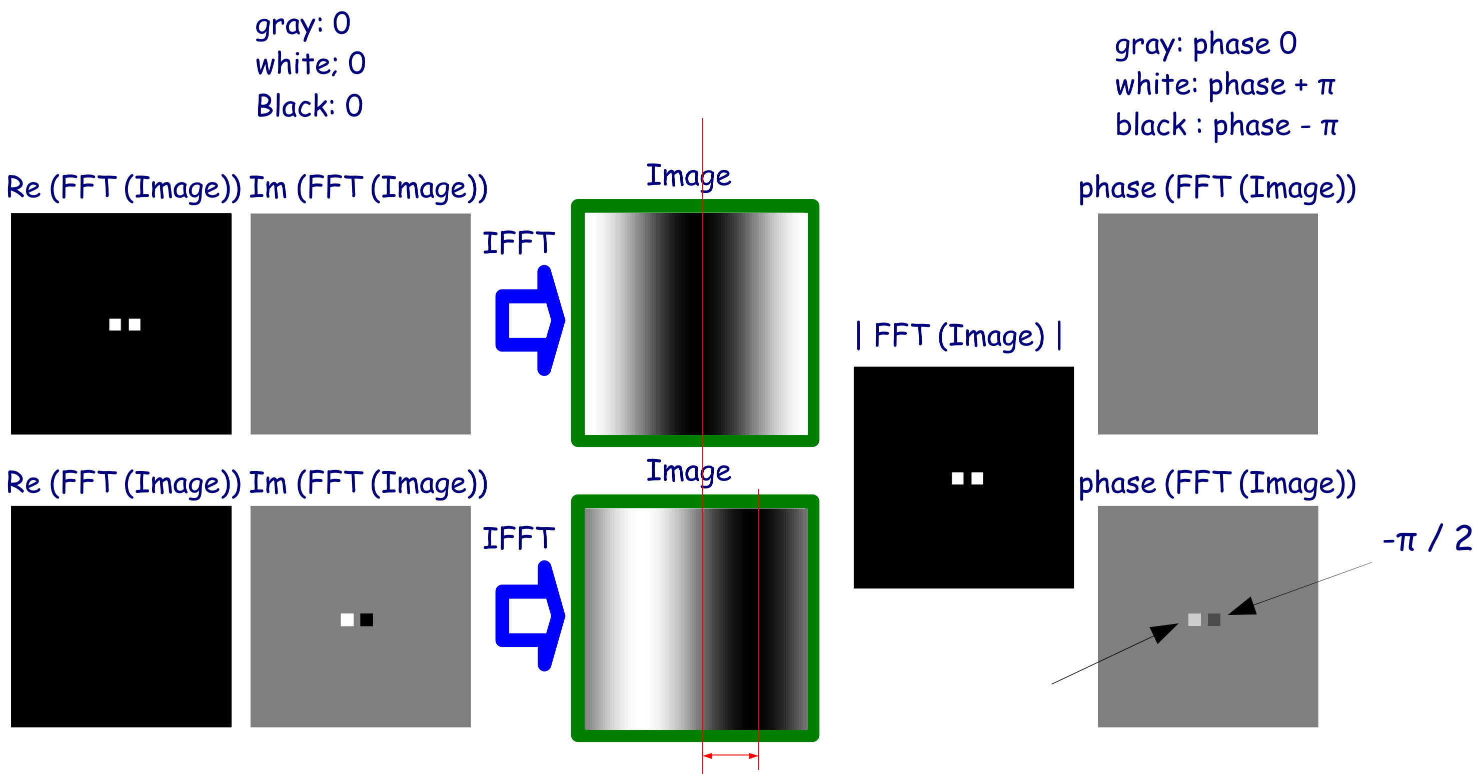
- basis functions are sines wave and cosine 2D

The amplitude of the basis functions

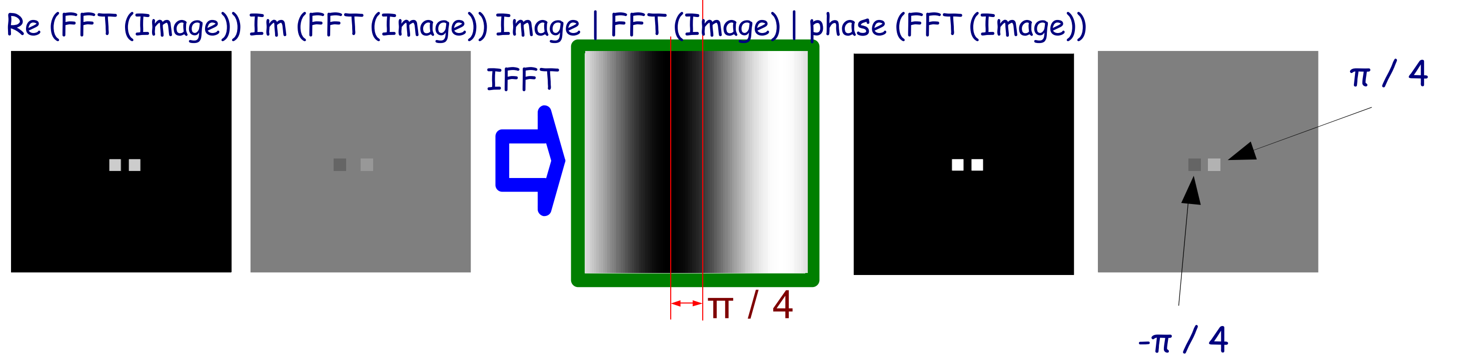


- height of the "peak" controls the amplitude (brightness) basis functions
- sine becomes less contrast - more gray (close to 0 - gray)

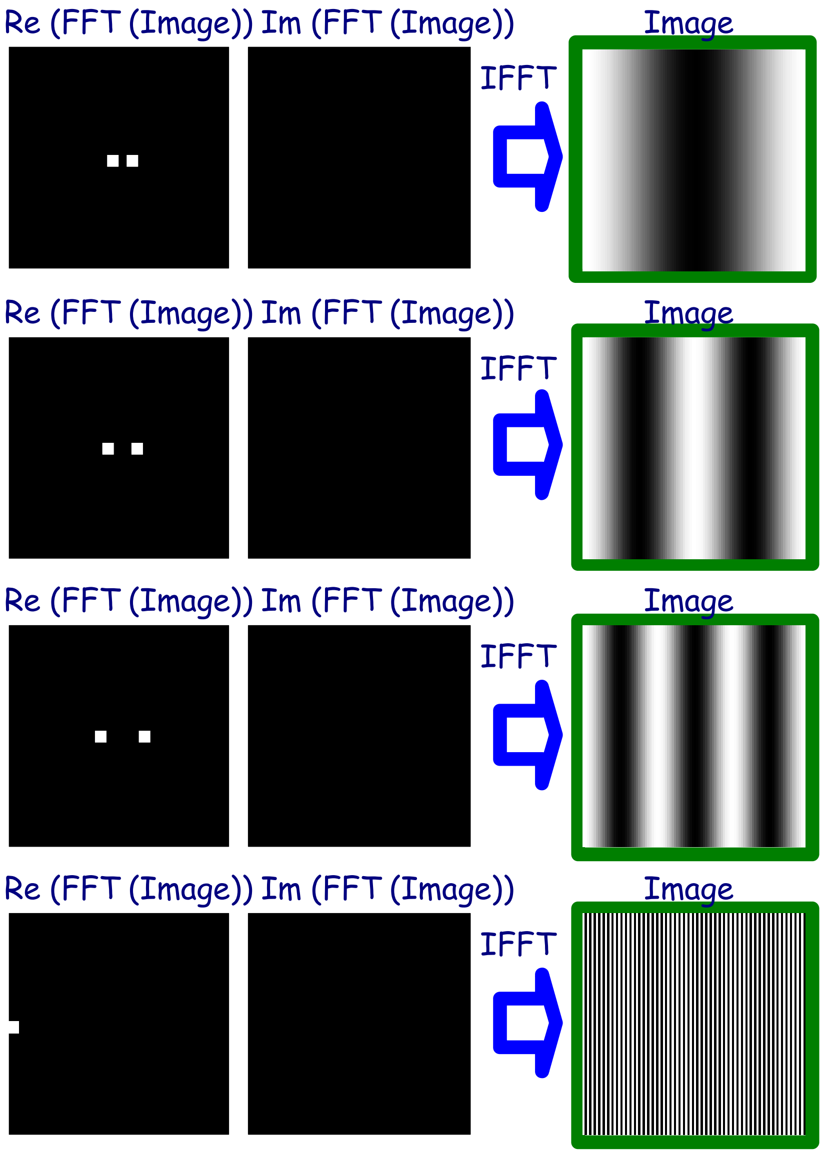
Phase of the basis functions



- phase controls shift basis functions (sine is the shifted phase cosine)



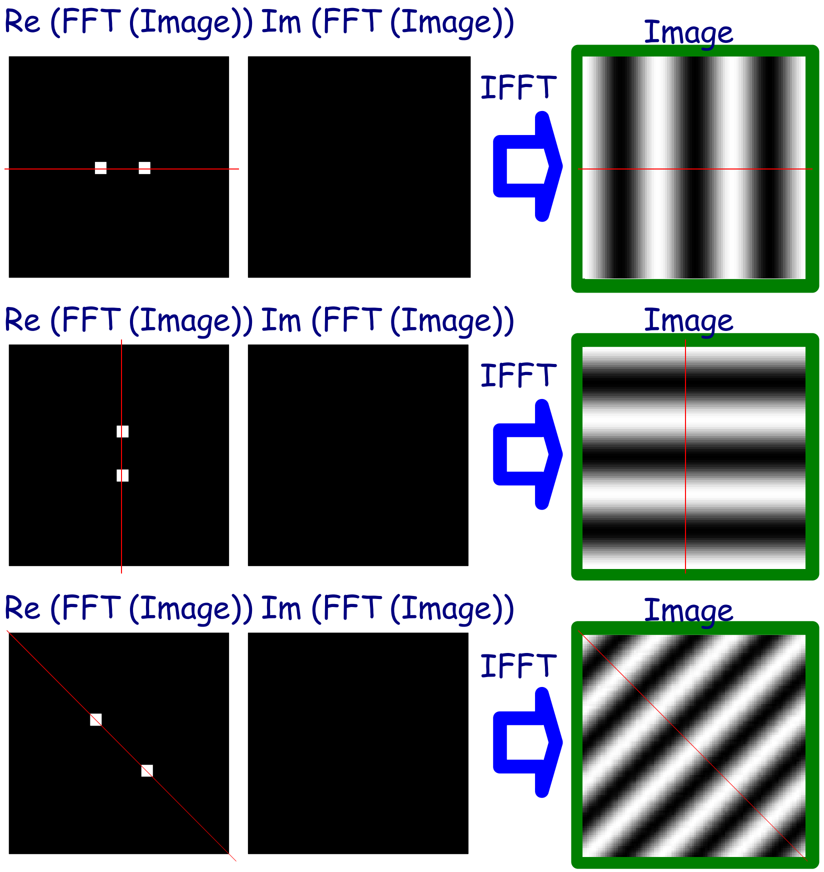
Frequency of basis functions



- position "peaks" controls the frequency basis functions
- closer to the center - lower frequencies
- away from the center - higher frequencies
- minimum frequency = 1
- maximum frequency = N / 2 (only one "peak" in FFT)

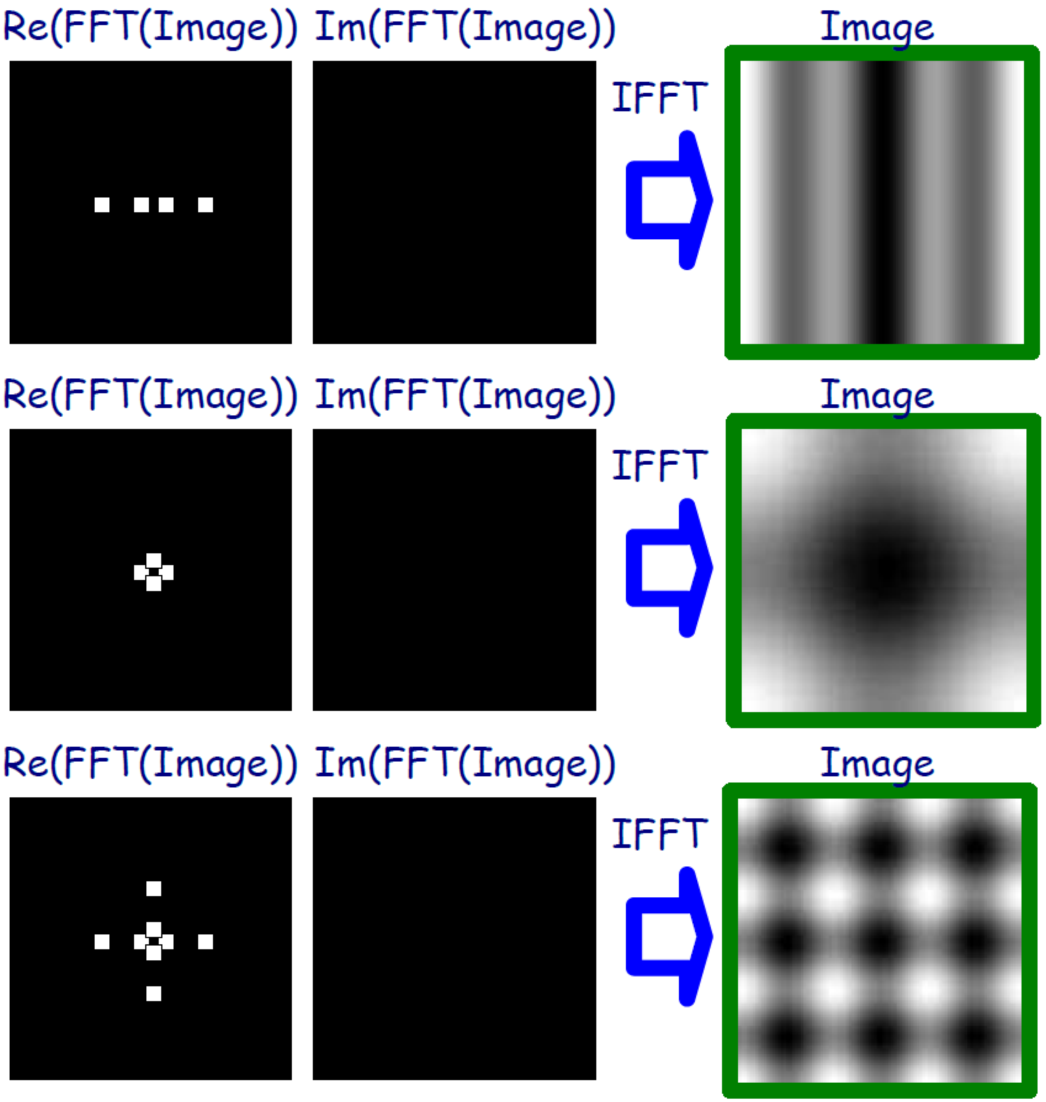
black and white stripes with a thickness of 1

Orientation basis functions



- the orientation of the basis functions is determined by the ratio of components of the x and y "peaks„ in the frequency domain

Submission of basis functions

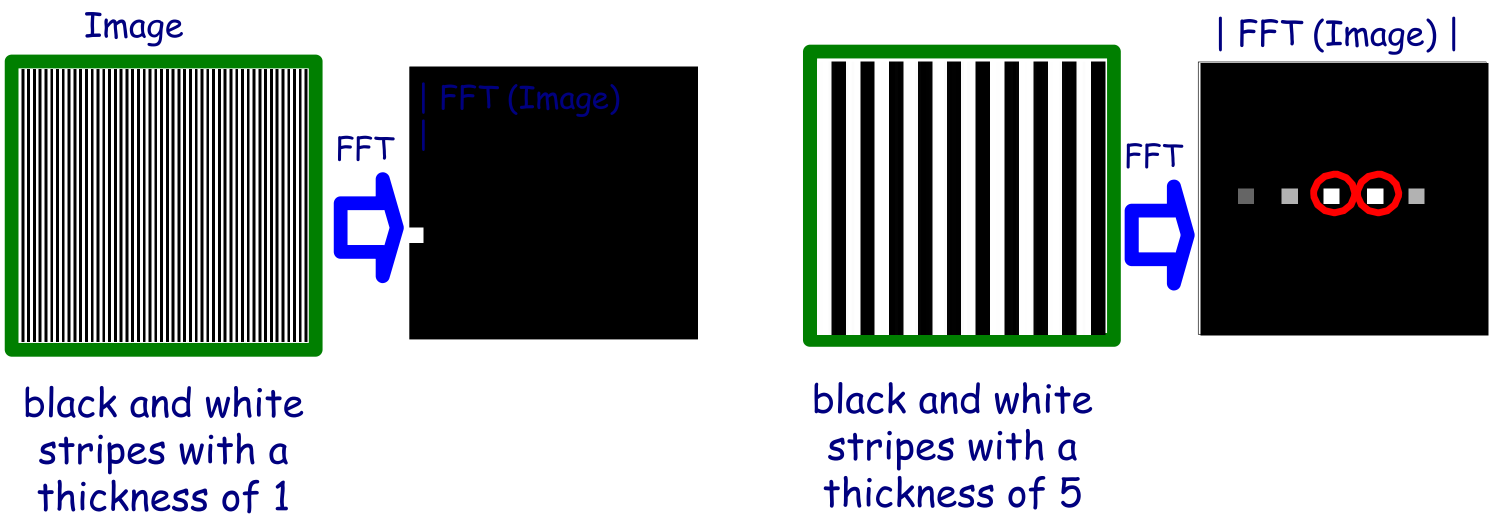


$$\cos(2\pi \cdot \frac{x}{N}) + \cos(2\pi \cdot \frac{3x}{N})$$

$$\cos(2\pi \cdot \frac{x}{N}) + \cos(2\pi \cdot \frac{y}{N})$$

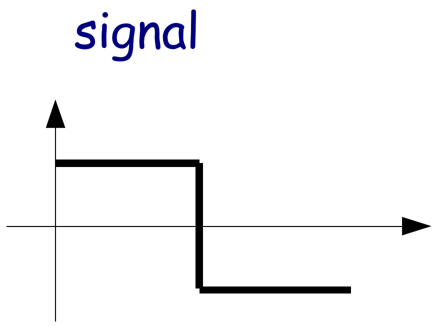
$$\cos(2\pi \cdot \frac{3x}{N}) + \cos(2\pi \cdot \frac{3y}{N})$$

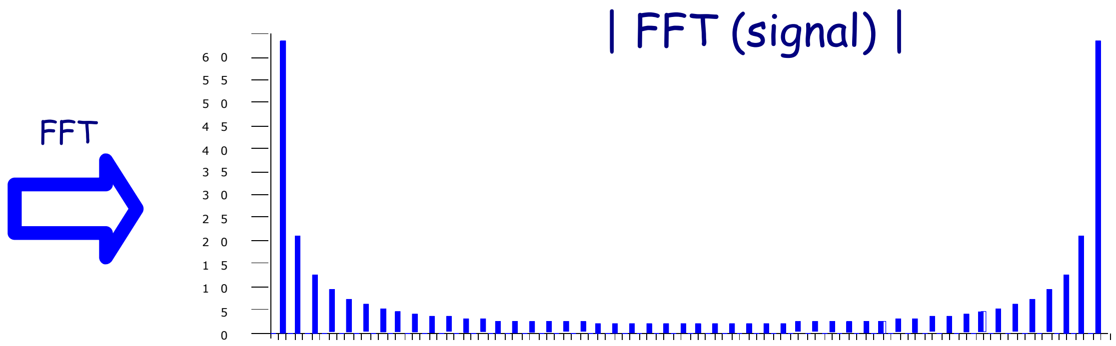
High-frequency transform



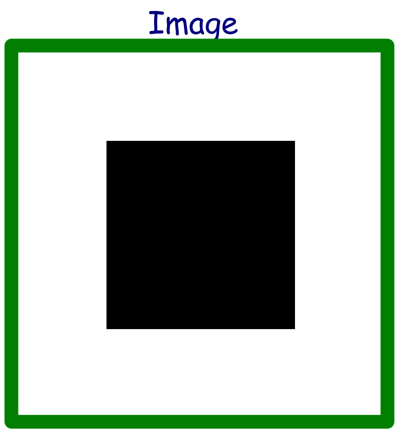
- the left image and its transform is understandable, the maximum frequency of the cosine - N / 2
- but the right?
- there are additional spectrum: in addition to the main (N / 10, circles) and its multiples: N / 5 and ~ N / 2 of lower amplitudes!
- That are component harmonics - this is because the sampled signal is no longer just a sinusoid, it is graph 0101010101

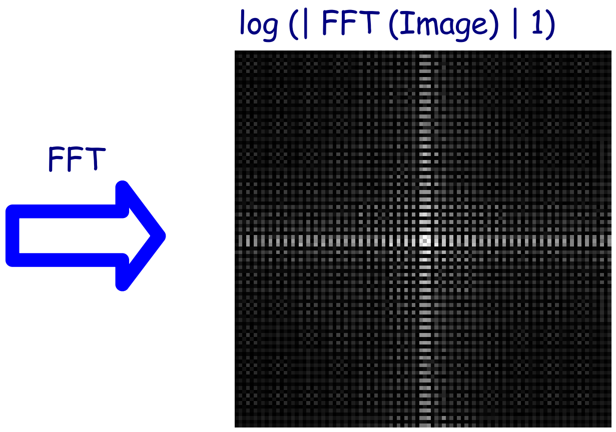
Transform of discrete signals



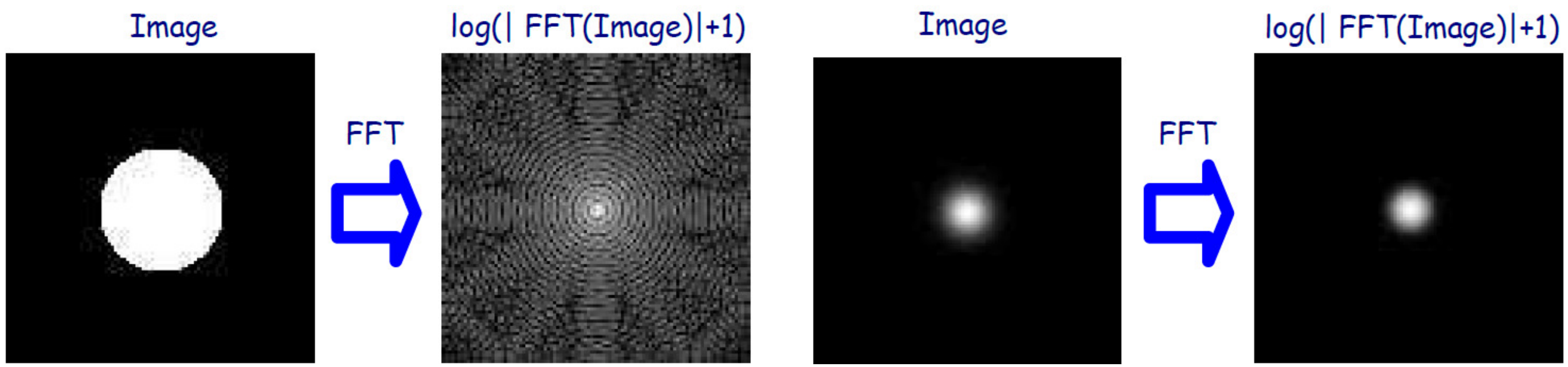


- Transform signal increments the value is a composite of many frequencies
- both in 2D and in 3D:

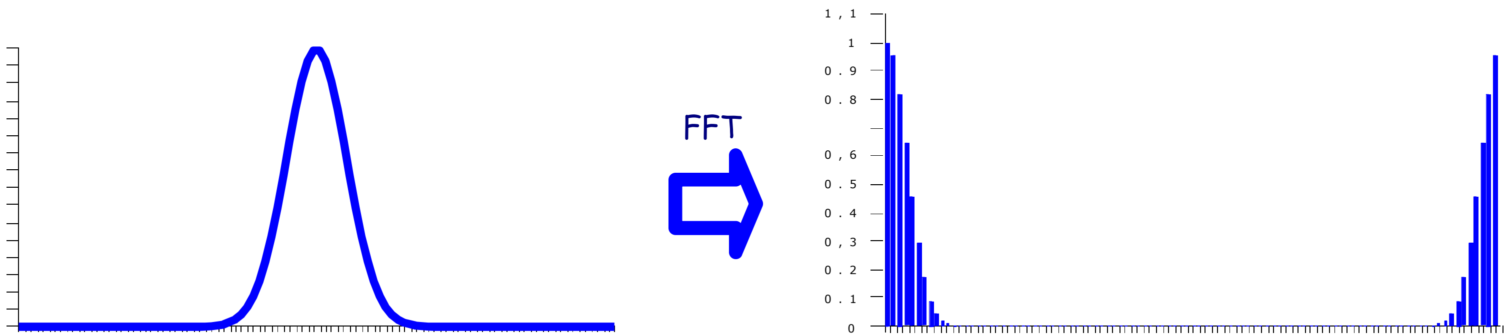




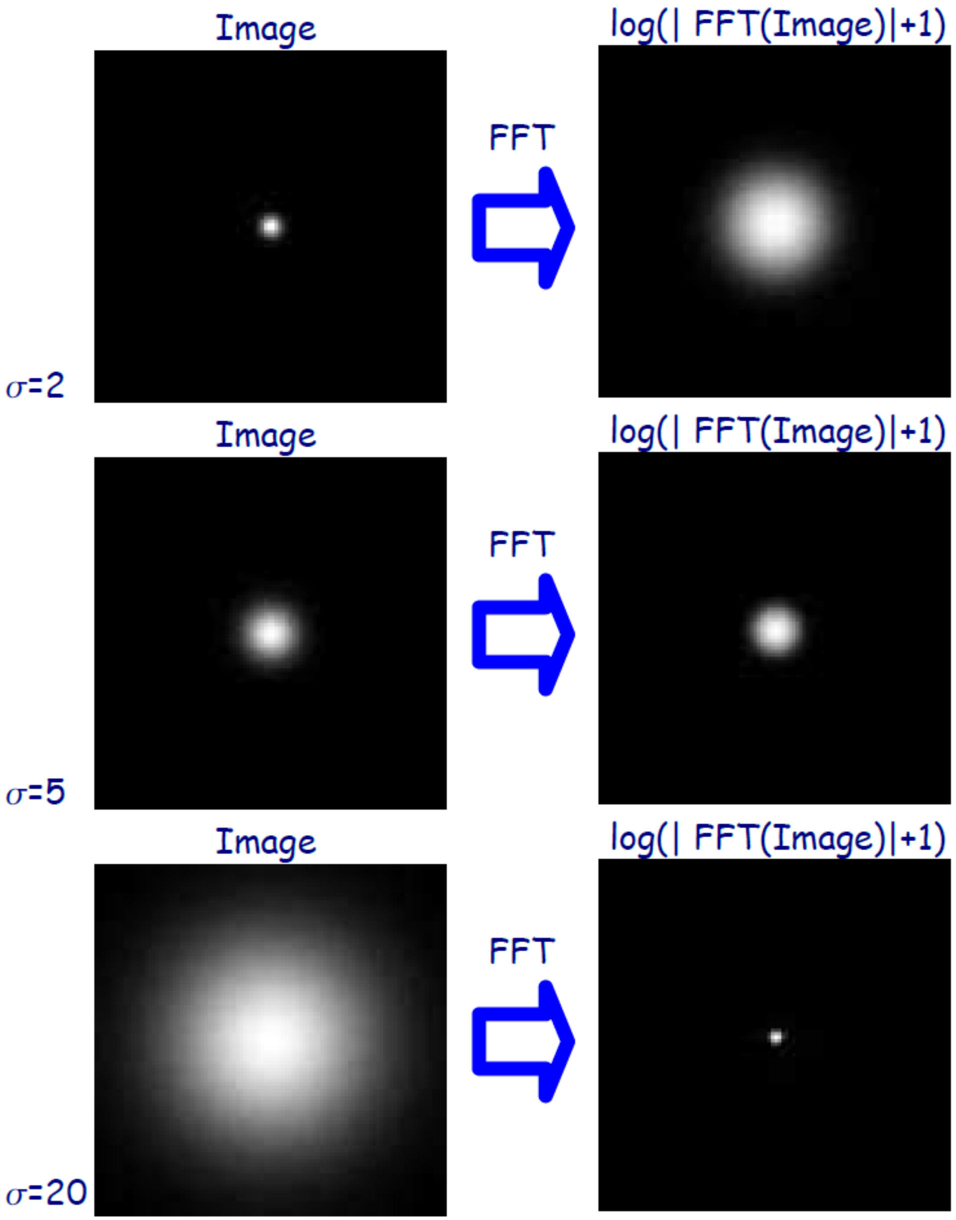
Transform Gaussian distribution



- Fourier transform Gaussian curve, is also a Gaussian curve!
- This is one of the few forms which are finite, compact shape in the frequency domain and the spatial domain
- as 3D and 2D:



The Heisenberg uncertainty principle



- can not have a small image of Gaussian functions in the frequency domain and spatial
- increasing the width in the spatial domain is reduced in the time domain
- the product of the width (standard deviation) in the spatial frequency domain and is a constant:

$$\sigma_{xy} \cdot \sigma_f = const$$

Image filtering: low-pass filter

- if we have the spectrum of the frequency of the image, you can select only the ones that interest us and remove the remaining
- for example, a smoothing filter is a low pass filter - stops in the image only the lower and higher frequencies: