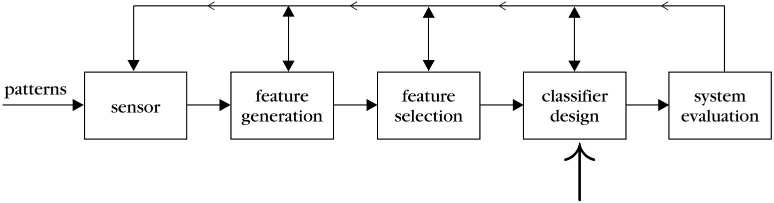
Pattern Recognition Lecture
“Linear Classifiers”

Prof. Dr. Marcin Grzegorzek

Research Group for Pattern Recognition
www.pr.informatik.uni-siegen.de

Institute for Vision and Graphics
University of Siegen, Germany

Pattern Recognition Chain



Overview

Overview

Introducing Example

Known

- A two-class problem $\Omega = \{\omega_1, \omega_2\}$ in a 2D feature space $\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$ is considered.
- The classifier is given by

$$y = 2x_1 + x_2$$

and

$$\begin{cases} y > 5 & \Rightarrow \quad i = 1 \\ y \leq 5 & \Rightarrow \quad i = 2 \end{cases}$$

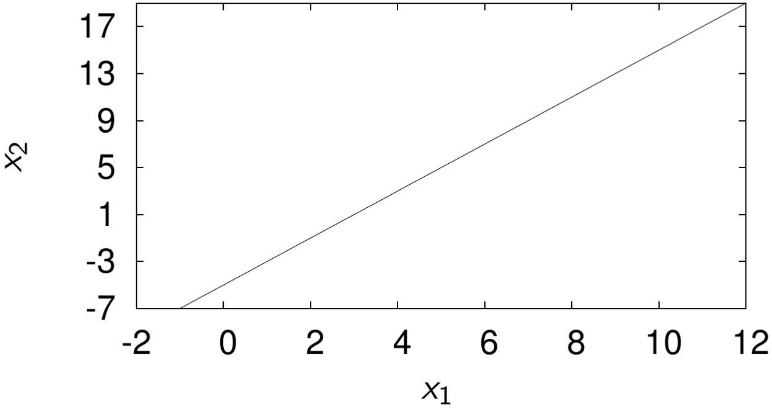
Task

- Find the decision line!

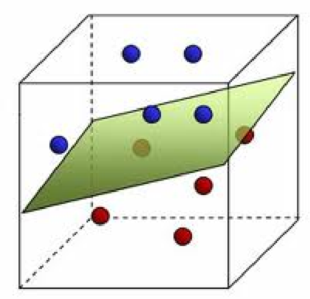
Solution

Yes, it is that simple as it sounds. The decision line is just given by

$$x_2 = 2x_1 - 5$$



Another Example for Linear Classification



Confusing Notation

Weight Vector without Threshold	Weight Vector with Threshold
$\mathbf{w} = [w_1, \ldots, w_l]^{\mathrm{T}}$	$\mathbf{w} = [w_1, \ldots, w_l, w_0]^{\mathrm{T}}$
$\mathbf{x} = [x_1, \ldots, x_l]^{\mathrm{T}}$	$\mathbf{x} = [x_1, \ldots, x_l, 1]^{\mathrm{T}}$
$\mathbf{w}^{\mathrm{T}}\mathbf{x} + w_0 = 0$	$\mathbf{w}^{\mathrm{T}}\mathbf{x} = 0$

Overview

Decision Hyperplanes for l-Dimensions (1)

- Let us focus on the two-class problem and consider linear discriminant functions. The decision hypersurface in the l-dimensional feature space is then given by

$$\mathbf{w}^{\mathrm{T}}\mathbf{x} = 0$$

- The dimensionality problem ($\mathbf{w} \in \mathbb{R}^{l+1}$, but feature vectors have l elements) is overcome by increasing the dimensionality of each feature vector, so that

$$\mathbf{x} = [x_1, x_2, \ldots, x_l, 1]^{\mathrm{T}}$$

This does not change anything in the linear classification process.

Decision Hyperplanes for l-Dimensions (2)

- If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two points on the decision hyperplane, then the following is valid

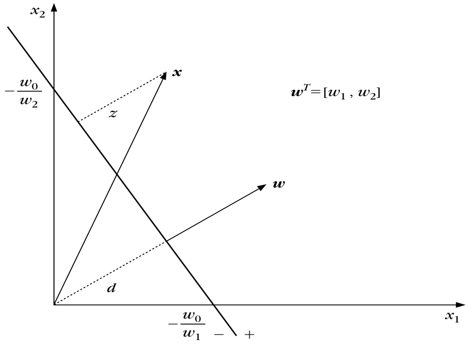
$$\mathbf{w}^{\mathrm{T}}\mathbf{x}_1 = \mathbf{w}^{\mathrm{T}}\mathbf{x}_2 = 0$$

$$\Updownarrow$$

$$\mathbf{w}^{\mathrm{T}}(\mathbf{x}_1 - \mathbf{x}_2) = 0$$

- Since the difference vector $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ obviously lies on the decision hyperplane, it is apparent that the weight vector $\mathbf{w}$ is orthogonal to the decision hyperplane.

Decision Hyperplanes for l-Dimensions (3)



$$d = \frac{|w_0|}{\sqrt{w_1^2 + w_2^2}} \qquad z = \frac{|g(\mathbf{x})|}{\sqrt{w_1^2 + w_2^2}}$$

Overview

Problem Statement

Problem

How to compute the unknown parameters $w_1, \dots, w_l, w_0$?

Assumptions

The two classes ω_1 and ω_2 are linearly separable, i. e., there exist a hyperplane $\widehat{\mathbf{w}}$ such that

$$\widehat{\mathbf{w}}^{\mathrm{T}}\mathbf{x} > 0; \qquad \forall \mathbf{x} \in \omega_1$$

$$\widehat{\mathbf{w}}^{\mathrm{T}}\mathbf{x} < 0; \qquad \forall \mathbf{x} \in \omega_2$$

Approach

The problem will be solved as an optimisation task. Therefore, we need:

- an appropriate cost function
- an algorithmic scheme to optimise it

Perceptron Cost Function - Definition

- As cost function the perceptron cost will be used:

$$J(\mathbf{w}) = \sum_{\mathbf{x} \in Y} (\delta_x \mathbf{w}^{\mathrm{T}} \mathbf{x})$$

- Y - subset of training vectors misclassified by the hyperplane $\mathbf{w}$

- The variable δ_x is chosen so that:

$$\begin{cases} \mathbf{x} \in \omega_1 & \Rightarrow & \delta_x = -1 \\ \mathbf{x} \in \omega_2 & \Rightarrow & \delta_x = +1 \end{cases}$$

Perceptron Cost Function - Properties

- The perceptron cost is not negative. It becomes zero when $Y = \emptyset$, that is, if there are no misclassified vectors $\mathbf{x}$
- Indeed, if $\mathbf{x} \in \omega_1$ and it is misclassified, then $\mathbf{w}^{\mathrm{T}}\mathbf{x} < 0$ and $\delta_x < 0$. Thus, the product is positive
- The perceptron cost function is continuous and piecewise linear

Minimisation of the Perceptron Cost Function (1)

- The iterative minimisation works according to:

$$\mathbf{w}(t+1) = \mathbf{w}(t) - \rho_t \left. \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} \right|_{\mathbf{w}=\mathbf{w}(t)}$$

- $\mathbf{w}$ is the weight vector at the iteration step no. t
- ρ_t is a positive real number chosen manually.

Minimisation of the Perceptron Cost Function (2)

- From the perceptron definition and the points where this is valid, we get

$$\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = \sum_{\mathbf{x} \in Y} \delta_x \mathbf{x}$$

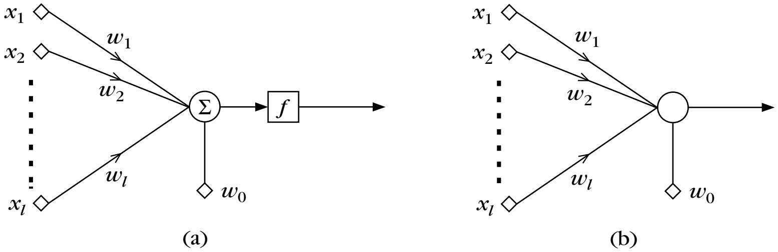
- Thus, the iterative minimisation of the cost function from the previous slide can be written as

$$\mathbf{w}(t+1) = \mathbf{w}(t) - \rho_t \sum_{\mathbf{x} \in Y} \delta_x \mathbf{x}$$

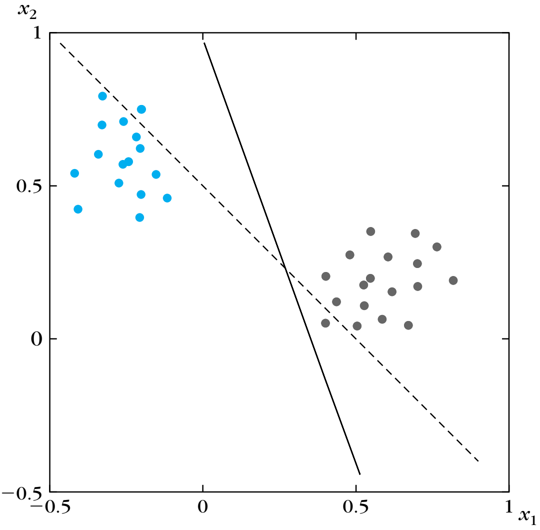
The Perceptron Algorithm - Pseudocode

- Choose $\mathbf{w}(0)$ randomly
- Choose ρ_0
- $t = 0$
- Repeat
  - Set $Y = \emptyset$
  - For $j = 1$ to K
    - If $\delta_{x_j}\mathbf{w}(j)^{\mathrm{T}}\mathbf{x}_j \geq 0$ then $Y = Y \cup \{\mathbf{x}_j\}$
  - End For
  - $\mathbf{w}(t+1) = \mathbf{w}(t) - \rho_t \sum_{\mathbf{x}\in Y} \delta_x \mathbf{x}$
  - Adjust ρ_t
  - Iterate $t = t + 1$
- Until $Y = \emptyset$

The Basic Perceptron Model



Example for the Perceptron Algorithm (1)



Example for the Perceptron Algorithm (2)

Known

- Decision line after the iteration no. t is given by

$$x_1 + x_2 - 0.5 = 0 \quad \Leftrightarrow \quad \mathbf{w}(t) = [1, 1, -0.5]^{\mathrm{T}}$$

- With $\rho_t = 0.7$
- Vectors misclassified: $[0.4, 0.05]^{\mathrm{T}}$ and $[-0.2, 0.75]^{\mathrm{T}}$

Unknown

- The decision line after the iteration no. $t + 1$:

$$\mathbf{w}(t+1) = \begin{bmatrix} w_1(t+1) \\ w_2(t+1) \\ w_0(t+1) \end{bmatrix} = ?$$

Example for the Perceptron Algorithm (3)

$$\mathbf{w}(t+1) = \begin{bmatrix} 1 \\ 1 \\ -0.5 \end{bmatrix} - 0.7(-1) \begin{bmatrix} 0.4 \\ 0.05 \\ 1 \end{bmatrix} - 0.7(+1) \begin{bmatrix} -0.2 \\ 0.75 \\ 1 \end{bmatrix}$$

$$\Updownarrow$$

$$\mathbf{w}(t+1) = \begin{bmatrix} 1.42 \\ 0.51 \\ -0.5 \end{bmatrix}$$

Note that the dimensionality of the misclassified vectors has been increased by one!

Overview

Mean Square Error Estimation

- Linear classifiers are fast, thus, they sometimes are applied even for classes that are not linearly separable.
- In this case, the desired output of a classifier $y(\mathbf{x}) = y$ is sometimes not equal to the real output $\mathbf{w}^{\mathrm{T}}\mathbf{x}$.
- The cost function expresses the mean square error (MSE) between the desired and the true outputs

$$J(\mathbf{w}) = E[|y - \mathbf{x}^{\mathrm{T}}\mathbf{w}|^2]$$

- To find the optimal separating hyperplane $\widehat{\mathbf{w}}$, the cost function is minimised with regard to $\mathbf{w} = [w_1, \ldots, w_l, w_0]^{\mathrm{T}}$

$$\widehat{\mathbf{w}} = \underset{\mathbf{w}}{\operatorname{argmin}}\, J(\mathbf{w})$$

Sum of Error Squares Estimation (1)

- Two-class problem with not separable classes is considered.
- The cost function here is the sum of error squares

$$J(\mathbf{w}) = \sum_{i=1}^{N} (y_i - \mathbf{x}_i^{\mathrm{T}} \mathbf{w})^2$$

- $y_i \in \{-1, 1\}$ is the desired output of the classifier for $\mathbf{x}_i$
- $\mathbf{x}_i^{\mathrm{T}} \mathbf{w}$ is the real output of the classifier for $\mathbf{x}_i$
- In order to find the optimal separating hyperplane $\widehat{\mathbf{w}}$, the cost function has to be minimised with respect to $\mathbf{w}$

$$\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{N} \mathbf{x}_i (y_i - \mathbf{x}_i^{\mathrm{T}} \widehat{\mathbf{w}}) = 0 \qquad (1)$$

Sum of Error Squares Estimation (2)

- The minimisation term (1) can be rewritten as follows:

$$\left(\sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}}\right) \widehat{\mathbf{w}} = \sum_{i=1}^{N} (\mathbf{x}_i y_i) \tag{2}$$

- For the sake of formulation let us define

$$X = \begin{bmatrix} \mathbf{x}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{x}_N^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} x_{1,1} & \dots & x_{1,l} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{N,1} & \dots & x_{N,l} & 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \tag{3}$$

- X contains all training feature vectors for both classes, and $\mathbf{y}$ is a vector consisting of the corresponding desired responses $y_i \in \{-1, 1\}$.

Sum of Error Squares Estimation (3)

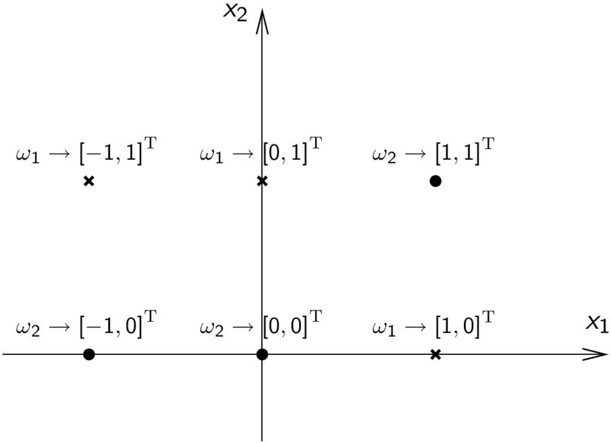
- Using both, (2) and (3) the following is true

$$(X^{\mathrm{T}}X)\widehat{\mathbf{w}} = X^{\mathrm{T}}\mathbf{y}$$

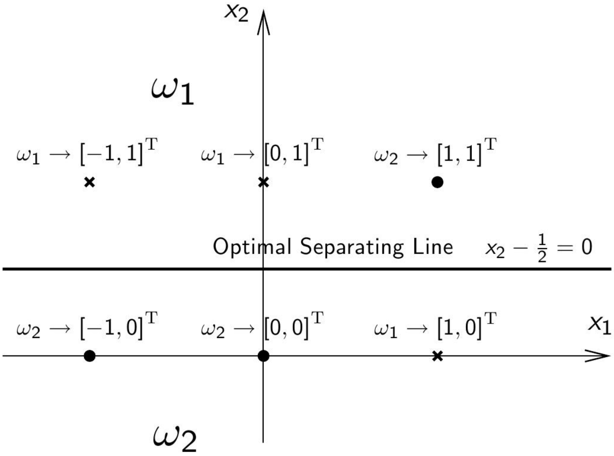
- Finally, the optimal separating hyperplane is given by

$$\widehat{\mathbf{w}} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\mathbf{y}$$

Sum of Error Squares Estimation - Example



Sum of Error Squares Estimation - Example



Overview

SVMs for Linearly Separable Classes (1)

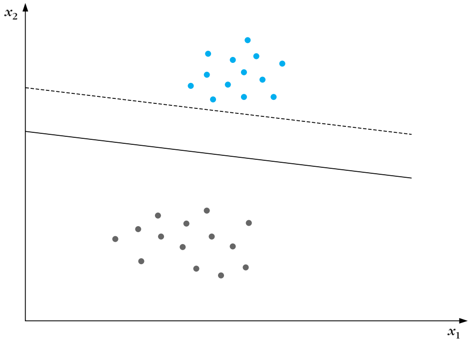
- A two-class problem $\Omega = \{\omega_1, \omega_2\}$
- $\mathbf{x}_{i=1,\ldots,N}$ are all training feature vectors
- The goal, once more, is to design a hyperplane[1]

$$g(\mathbf{x}) = \mathbf{w}^{\mathrm{T}}\mathbf{x} + w_0 = 0$$

that classifies correctly all the training feature vectors.

[1] Note that $\mathbf{w} = [w_1, \ldots, w_l]^{\mathrm{T}}$ and w_0 are treated separately here.

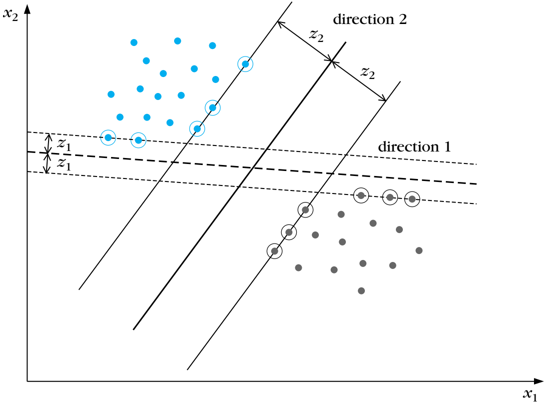
SVMs for Linearly Separable Classes (2)



- As we have seen for the perceptron algorithm, such a hyperplane is not unique.
- However, the full-line secures higher generalisation performance of the classifier, because it leaves the maximum margin from both classes.

SVMs for Linearly Separable Classes (3)

- The goal is to search for the direction that gives the maximum possible margin.



SVMs for Linearly Separable Classes (4)

- The distance of a point from a hyperplane is given by

$$z = \frac{|g(\mathbf{x})|}{||\mathbf{w}||}$$

- $\mathbf{w}$ and w_0 are now scaled so that the value $|g(\mathbf{x})|$ at the nearest points in both classes is equal to 1:

$$\begin{cases} \mathbf{w}^{\mathrm{T}}\mathbf{x} + w_0 \geq 1 & \forall \mathbf{x} \in \omega_1 \\ \mathbf{w}^{\mathrm{T}}\mathbf{x} + w_0 \leq -1 & \forall \mathbf{x} \in \omega_2 \end{cases}$$

- In this case, the margin is equal to

$$\frac{1}{||\mathbf{w}||} + \frac{1}{||\mathbf{w}||} = \frac{2}{||\mathbf{w}||}$$

SVMs for Linearly Separable Classes (5)

- In order to make the margin maximum, the following cost function has to be minimised

$$J(\mathbf{w}, w_0) = \frac{1}{2}||\mathbf{w}||^2$$

subject to

$$y_i(\mathbf{w}^{\mathrm{T}}\mathbf{x}_i + w_0) \geq 1; \quad \forall i = 1, 2, \ldots, N$$

SVMs for Linearly Separable Classes (6)

- Using the so called Lagrange function $\mathcal{L}(\mathbf{w}, w_0, \boldsymbol{\lambda})$ the Karush-Kuhn-Tucker (KKT) conditions have to be satisfied to minimise the cost function

(i) $\frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\mathbf{w}, w_0, \boldsymbol{\lambda}) = \mathbf{0}$

(ii) $\frac{\partial}{\partial w_0} \mathcal{L}(\mathbf{w}, w_0, \boldsymbol{\lambda}) = 0$

(iii) $\lambda_i \geq 0; \quad \forall i = 1, \ldots, N$

(iv) $\lambda_i [y_i(\mathbf{w}^{\mathrm{T}} \mathbf{x}_i + w_0) - 1] = 0; \quad \forall i = 1, \ldots, N$

SVMs for Linearly Separable Classes (7)

- The Lagrange function itself is defined as

$$\mathcal{L}(\mathbf{w}, w_0, \boldsymbol{\lambda}) = \frac{1}{2}\mathbf{w}^{\mathrm{T}}\mathbf{w} - \sum_{i=1}^{N} \lambda_i [y_i(\mathbf{w}^{\mathrm{T}}\mathbf{x}_i + w_0) - 1]$$

- Applying the KKT criteria (i) and (ii) for the Lagrange function

$$\mathbf{w} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i$$

$$\sum_{i=1}^{N} \lambda_i y_i = 0$$

SVMs for Linearly Separable Classes - Discussion

The Lagrange multipliers can be either zero or positive. Thus, the vector **w** of the optimal solution is a linear combination of $N_s \leq N$ feature vectors that are associated with $\lambda_i \neq 0$.

$$\mathbf{w} = \sum_{i=1}^{N_s} \lambda_i y_i \mathbf{x}_i$$

These are known as **support vectors** and the optimum hyperplane classifier as **support vector machine**.