Introduction to Mathematical Morphology

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Outline

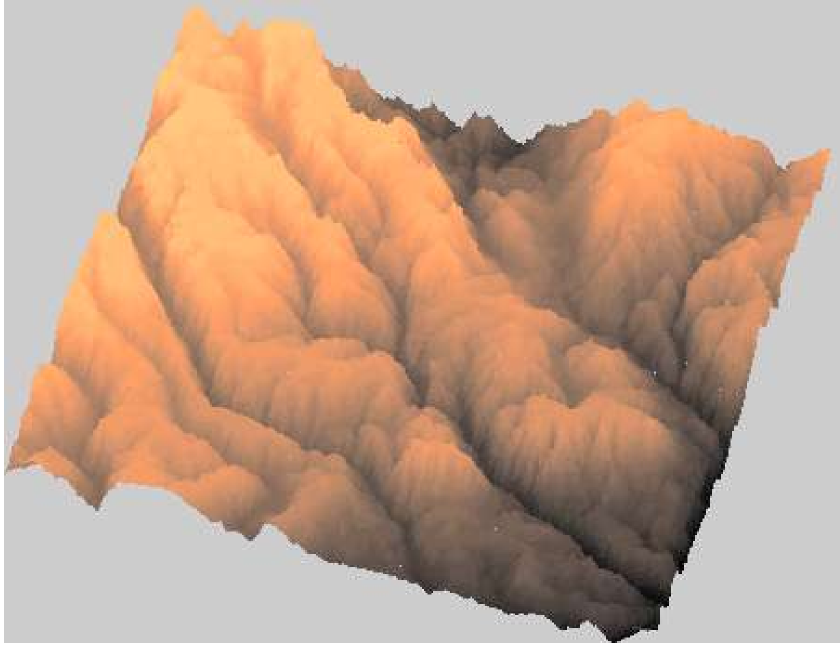
- Introduction
- Mathematical background
- Dilation and erosion
- Opening and closing
- Hit and miss transform, skeletons
- Geodesic dilation, erotion, and reconstruction
- Watershed segmentation

Mathematical Morphology - Introduction

- Based on shapes in the image, not pixel intensities
- Can be viewed as a general image processing framework
  - Various image processing techniques can be implemented by combining only a few simple operations
    - Examples: gradients, distance images, skeletons, noise removal, contrast enhancement, filling
- Typically used before and after an image segmentation
  - Exception: watershed segmentation
- All mathematical morphology operations are based on dilation and erosion
- The image processing toolkit in Matlab includes many mathematical morphology operations

Mathematical background

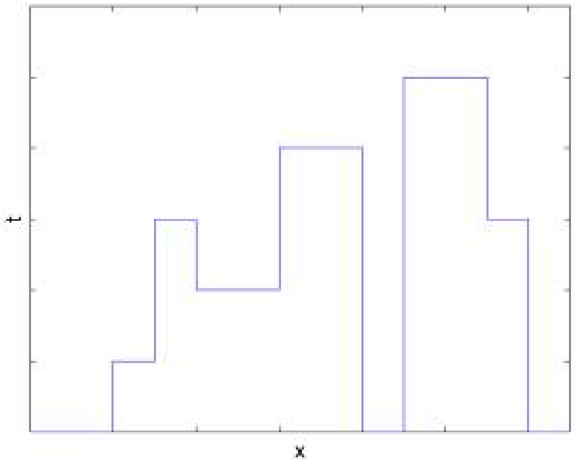
- In mathematical morphology we regard the pixel intensities as topographical highs



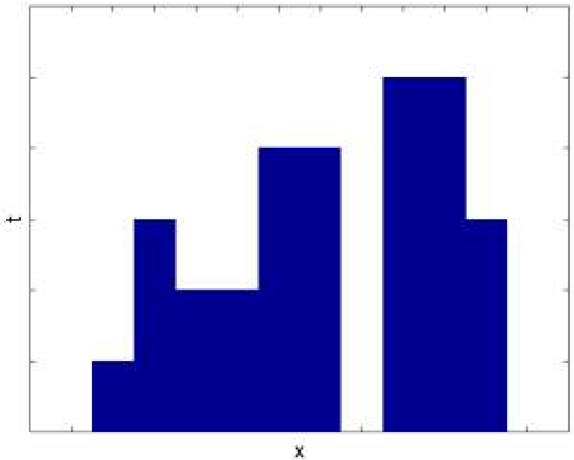
Mathematical background

- Set theory is used to define dilation and erosion
- We can transform an image into a set of points by regarding the image's subgraph
  - The subgraph of an image is defined as the set of points that lies between the graph of the image and above the image plane

$$SG(f)=\{(x,t)\in \mathbb{Z}^n * \mathbb{N} \mid 0 \leq t \leq f(x)\}$$



graph



subgraph

Erosion

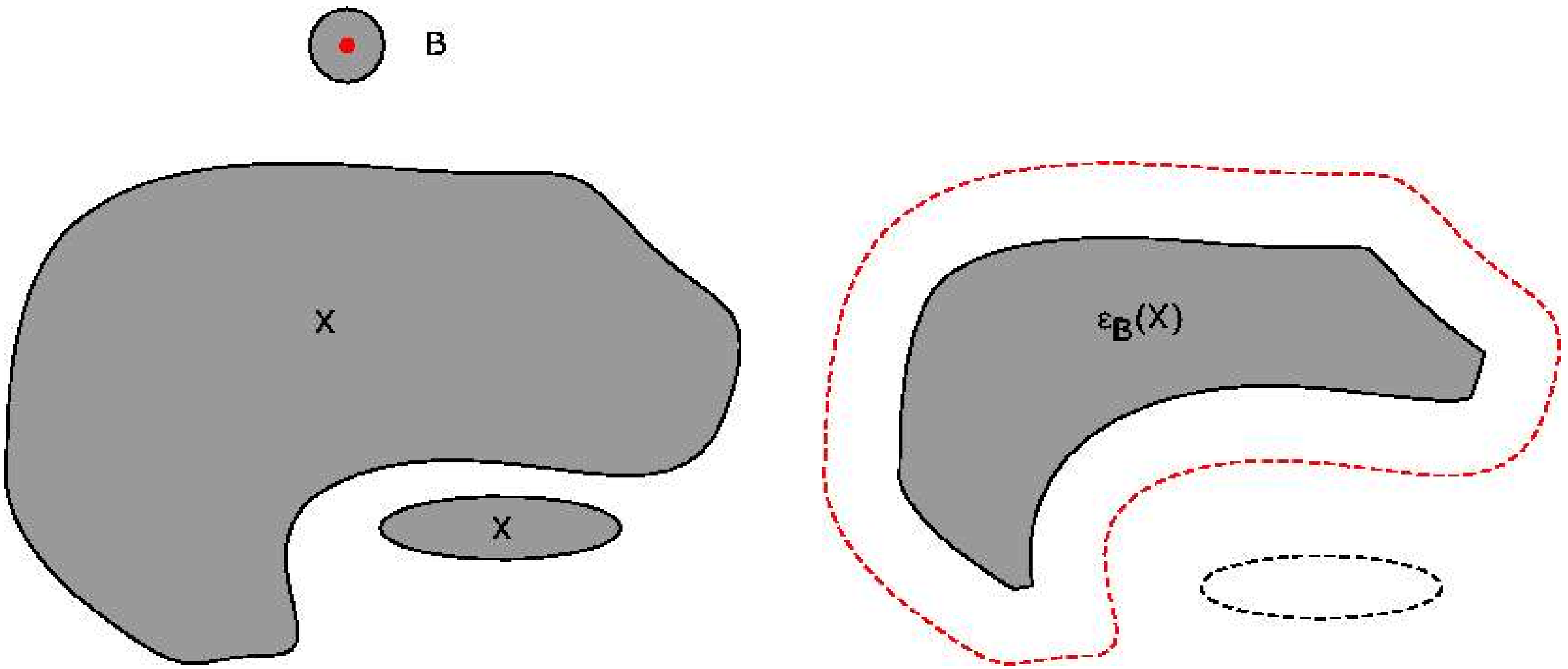
- Used to reduces objects in the image
- Definition, binary images:
  - The positions where a given structure element fits

    $$\varepsilon_B(X) = \{x \mid B_x \subseteq X\}$$

    where B_x means B translated with x,
    X is the image, and
    B is the structure element

Erosion

- Example, binary image



Erosion

- Definition, grayscale images
  We remember the definition for binary images:

  $$\varepsilon_B(X) = \{x \mid B_x \subseteq X\}$$

  Can be rewritten into the intersections of the translated sets X_{-b}:

  $$\varepsilon_B(X) = \cap_{b \in B} X_{-b}$$

  Which can be extended to also include grayscale images:

  $$\varepsilon_B(f) = \wedge_{b \in B} f_{-b}$$

Erosion

- Based on

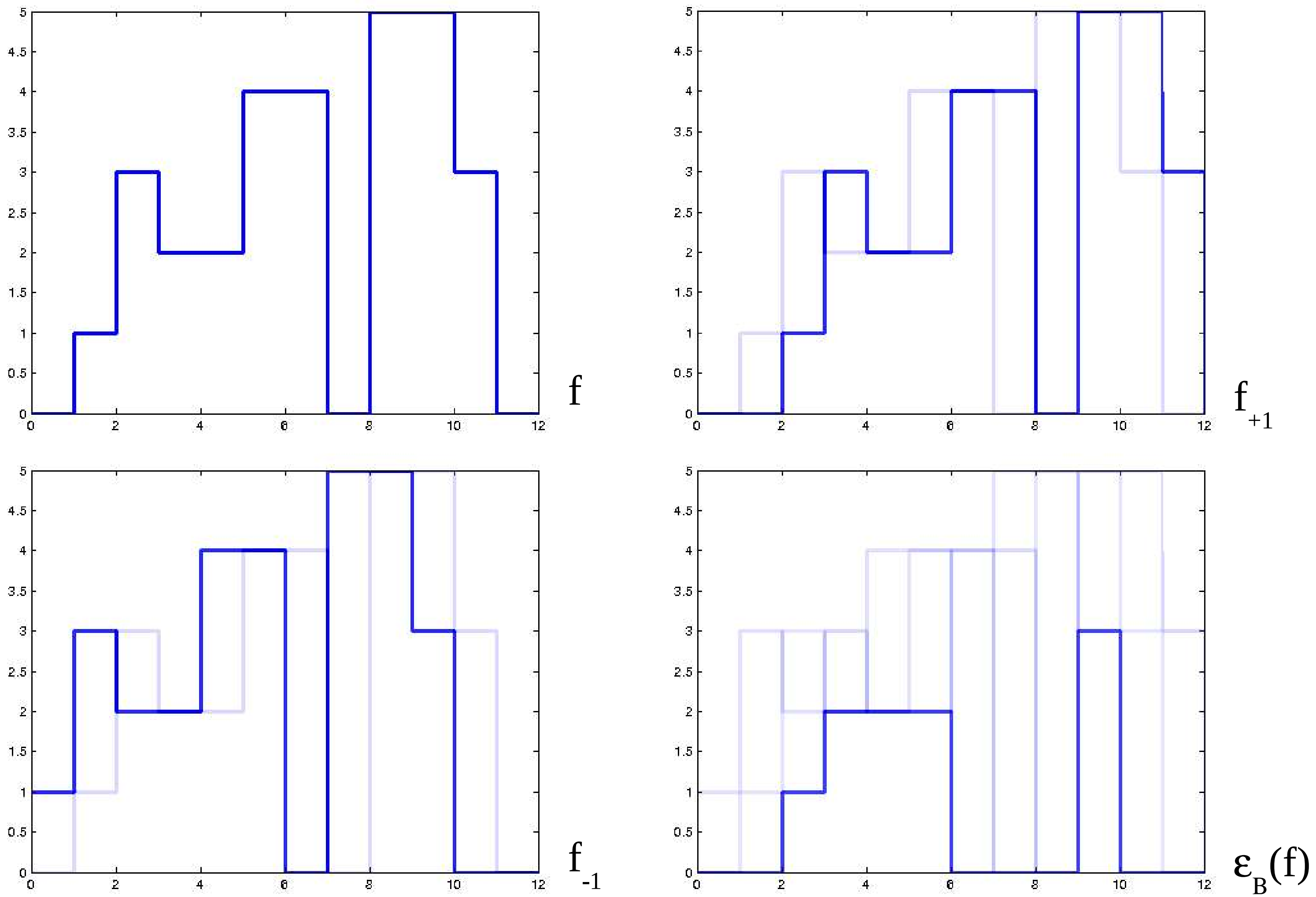
  $$\varepsilon_B(f) = \wedge_{b \in B} f_{-b}$$

  we can define an algorithm to find the erosion of image *f*:

  $$[\varepsilon_B(f)](x) = min_{b \in B} f(x+b)$$

Erosion

- Example, grayscale image



Dilation

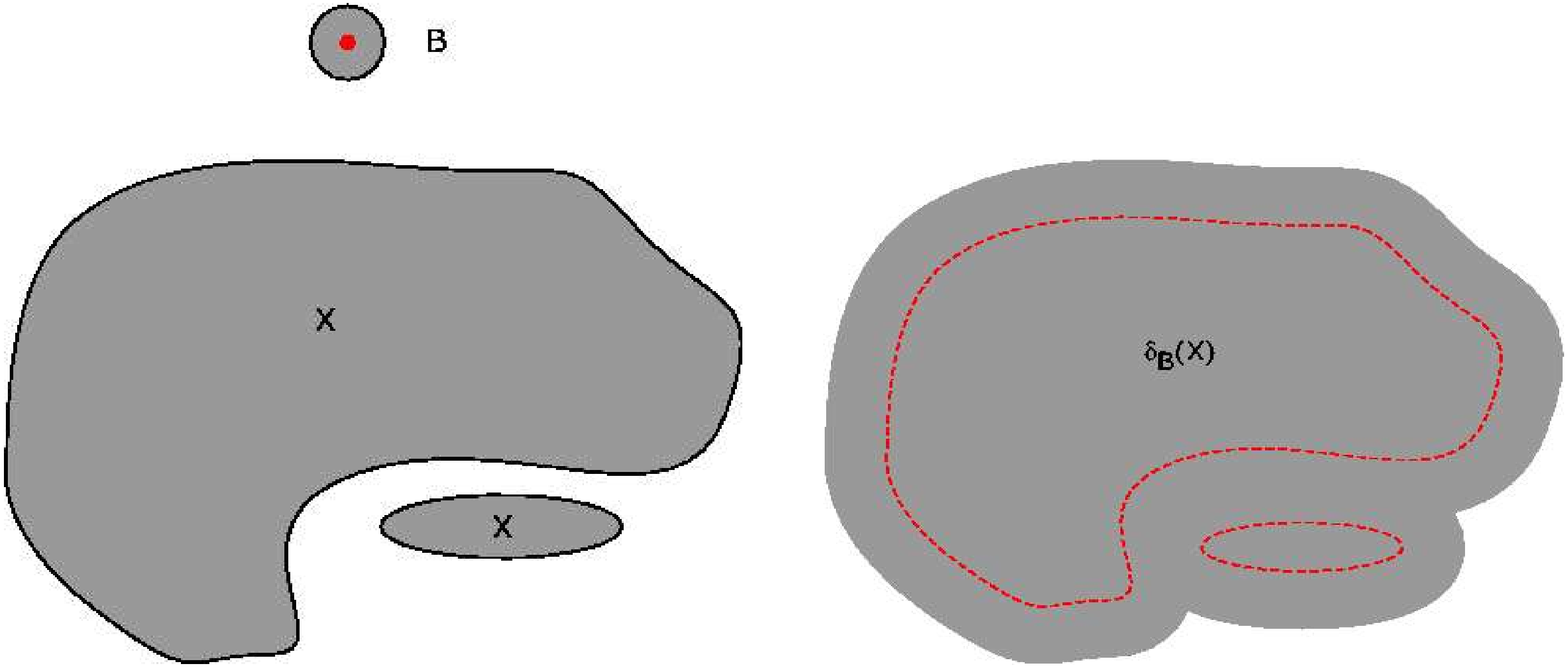
- Used to increase objects in the image
- Definition, binary images:
  - The positions where a given structure element fits

    $$\delta_B(X) = \{x \mid B_x \cap X \neq \emptyset\}$$

    where B_x means B translated with x,
    X is the image, and
    B is the structure element

Dilation

- Example, binary image



Dilation

- Definition, grayscale images
  We remember the definition for binary images:

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  Can be rewritten into the unions of the translated sets X_{-b}:

  $$\varepsilon_B(X) = \cup_{b \in B} X_{-b}$$

  Which can be extended to also include grayscale images:

  $$\varepsilon_B(f) = \vee_{b \in B} f_{-b}$$

Dilation

- Based on

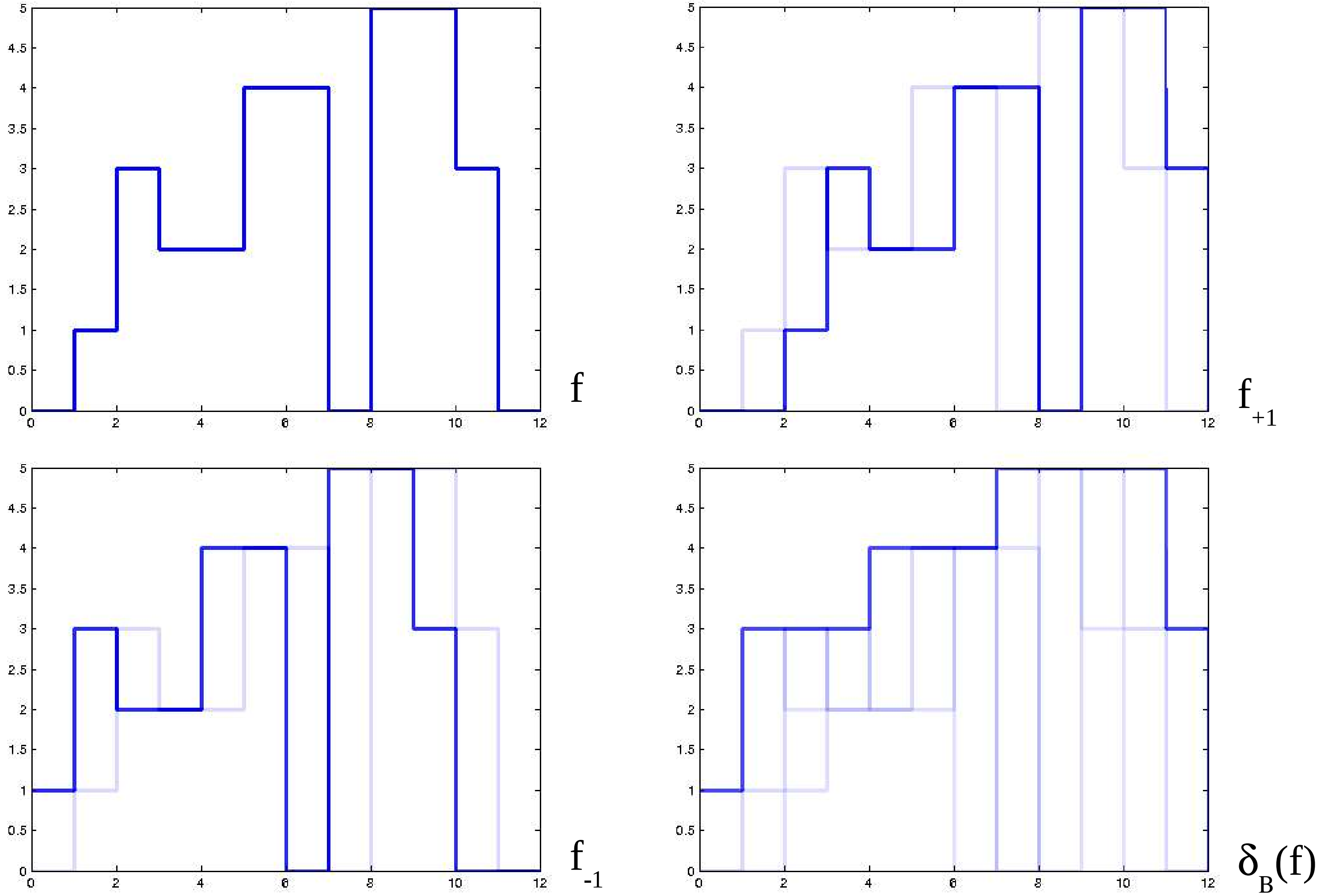
  $\varepsilon_B(f) = \vee_{b \in B} f_{-b}$

  we can define an algorithm to find the erosion of image *f*:

  $[\varepsilon_B(f)](x) = max_{b \in B} f(x+b)$

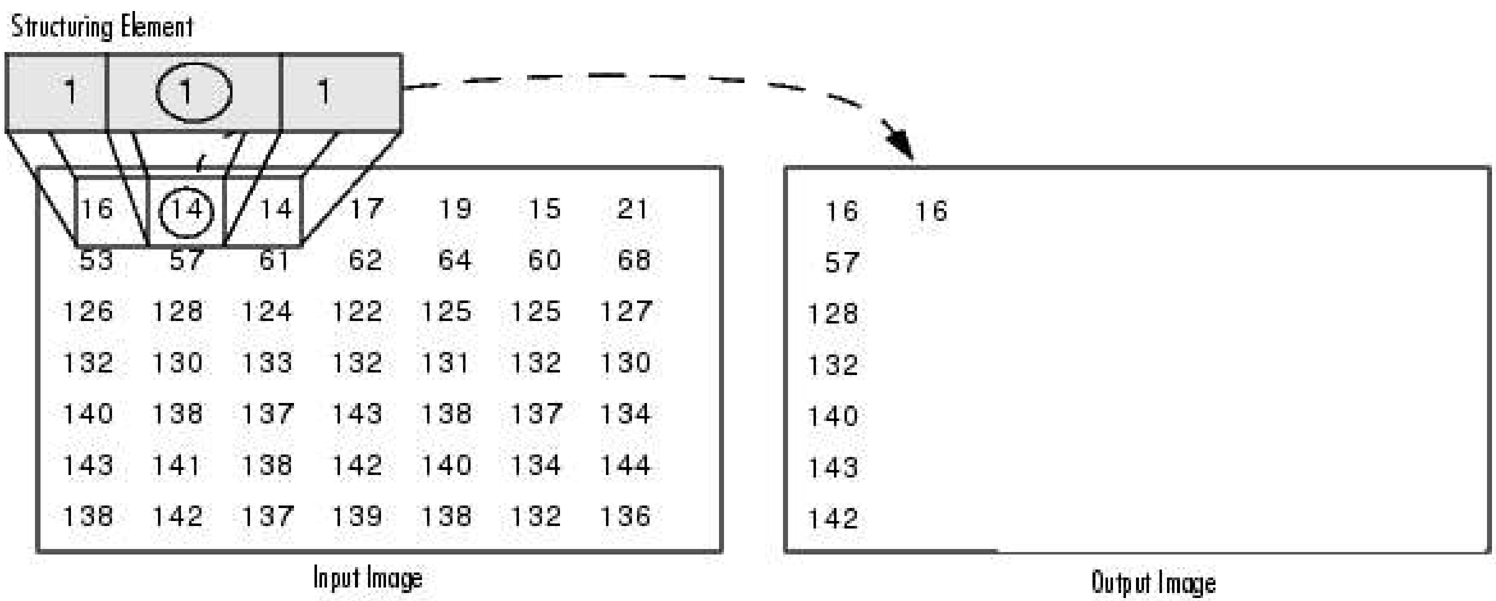
Dilation

- Example, grayscale image

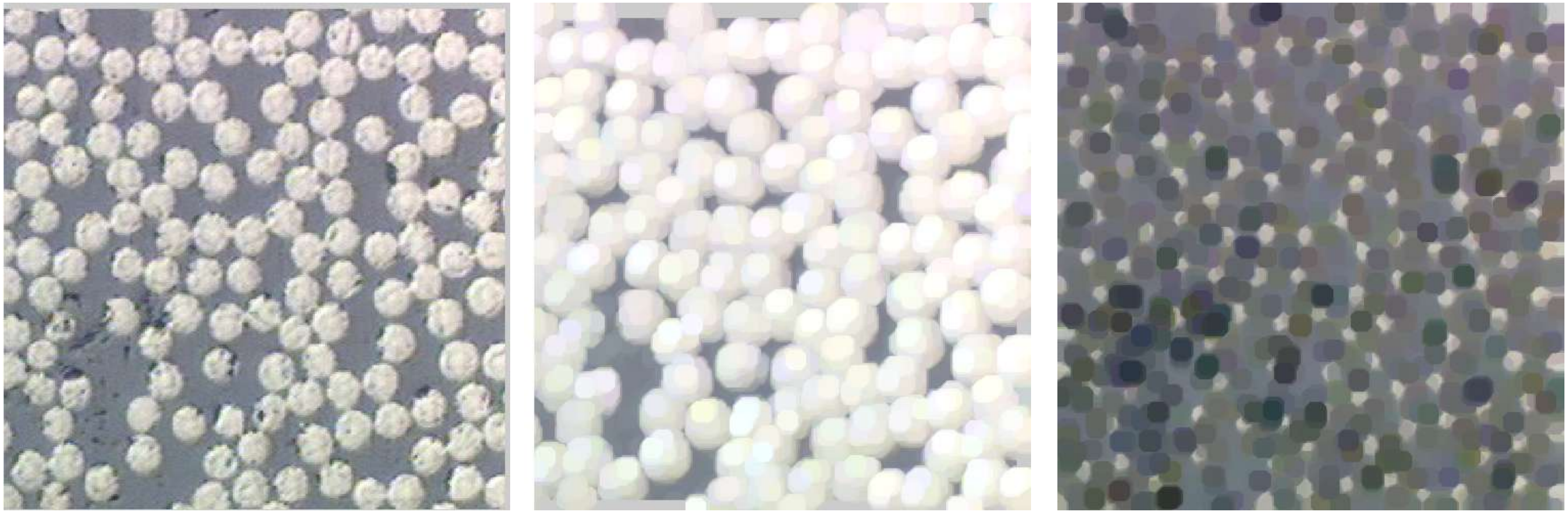


Dilation

- Example 2, grayscale image



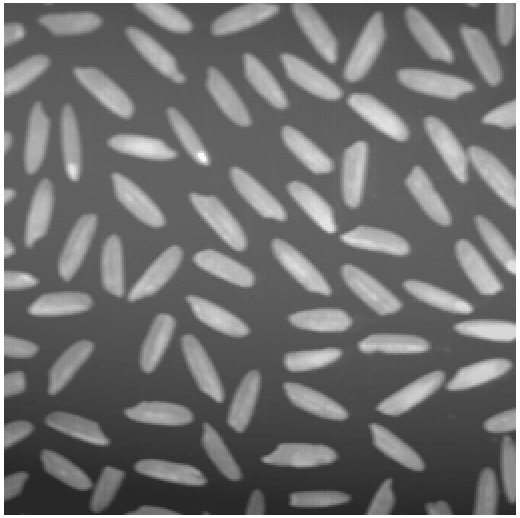
Dilation and erosion example

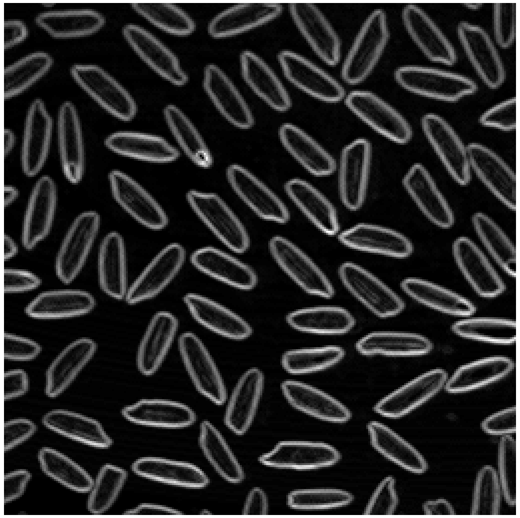


Beucher gradient

- Dilation and erosion can be used to extract edge information from images
  - Example: Beucher gradient

$$\rho_B = \delta_B - \varepsilon_B$$



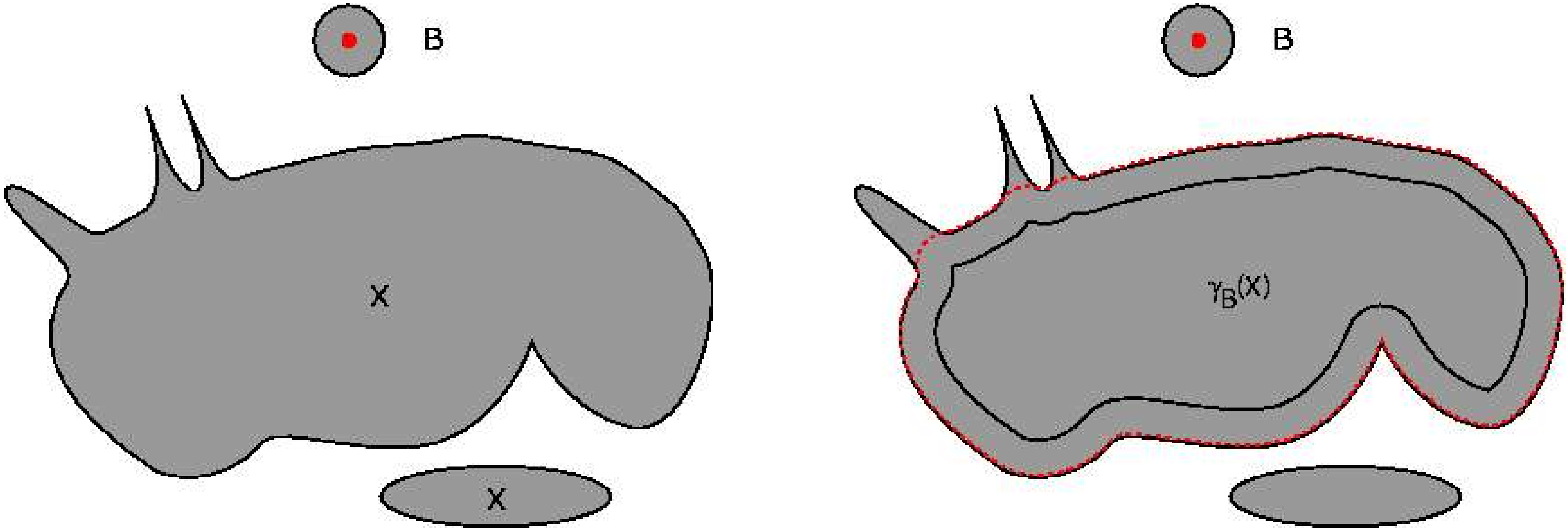


Morphological opening

- Used to remove unwanted structures in the image (e.g. noise)
- Morphological opening is simply an erosion followed by a dilation:

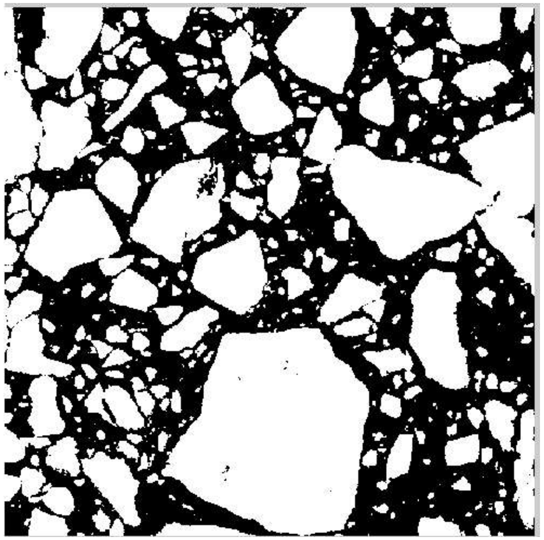
$$\gamma_B(f) = \delta_{\check{B}}[\varepsilon_B(f)]$$

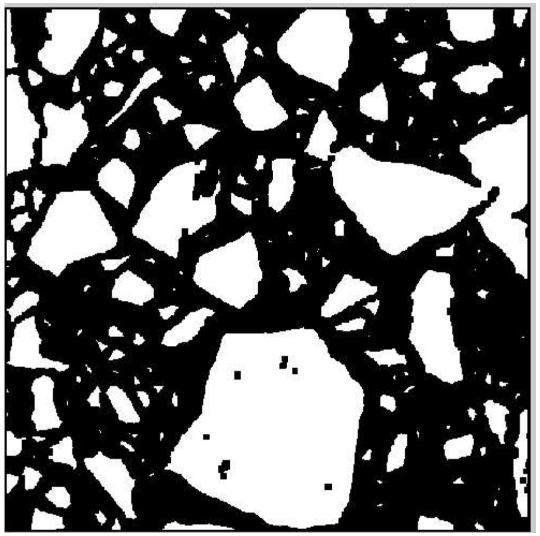
- Binary example:

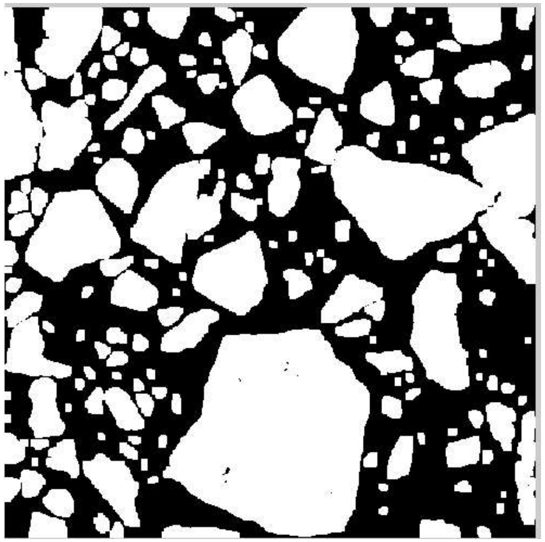


Morphological opening

- Grayscale example





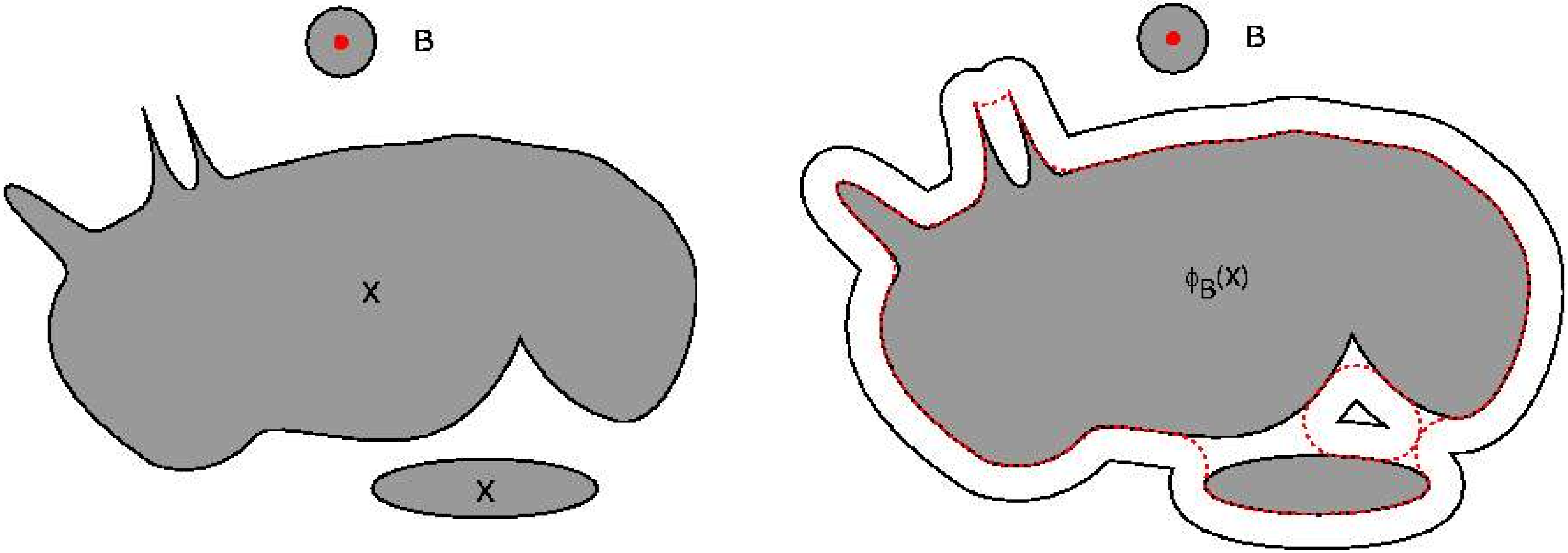


Morphological closing

- Is used to merge or fill structures in an image
- Morphological closing is dilation followed by erosion:

$$\phi_B(f) = \varepsilon_{\check{B}}[\delta_B(f)]$$

- Binary example:



Another example

- How to segment the text from the uneven illumination in the image?

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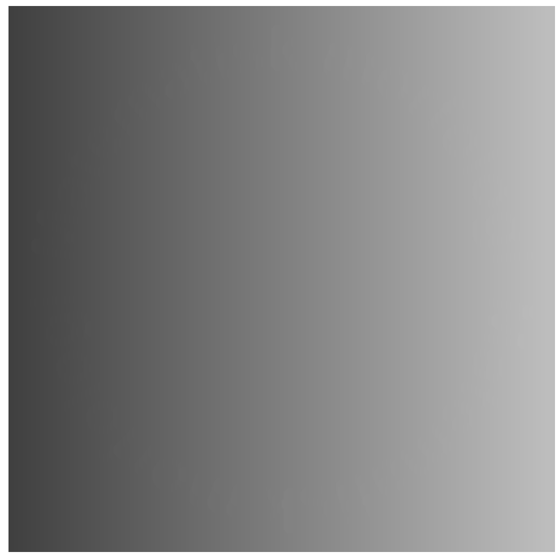
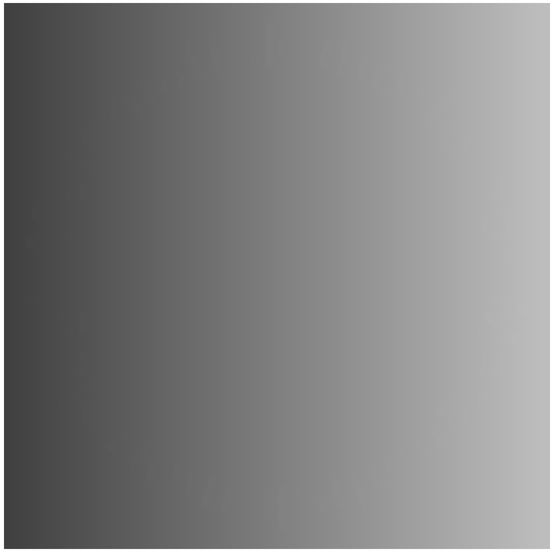


Another example

- How to segment the text from the uneven illumination in the image?

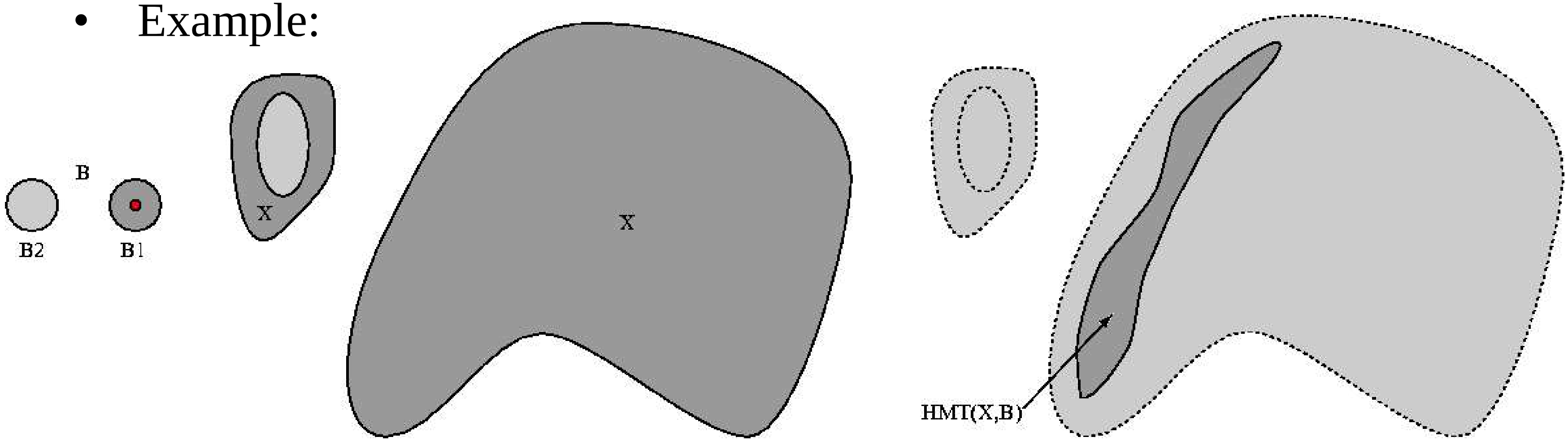


Hit and miss transform

- Used to extract pixels with specific neighbourhood configurations from an image
- Defined only for binary images
- Uses two structure elements B1 and B2 to find a given foreground and background configuration, respectively

$$HMT_B(X) = \{x \mid (B_1)_x \subseteq X, (B_2)_x \subseteq X^C\}$$

- Example:

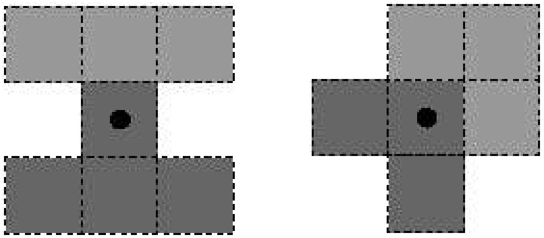


Thinning

- Used to shrink objects in binary images
- Differs from erosion in that objects are never completely removed
- Successive thinning until stability results in object skeletons
- Thinning is defined as:

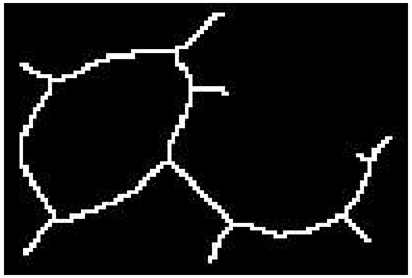
$$THIN(X,B) = X \setminus HMT_B(X)$$

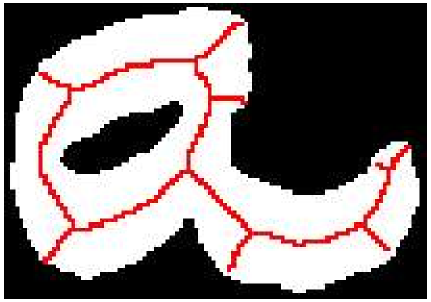
- Structure elements typically used in thinning (rotated 90 degrees 3 times to create 8 structure elements):



Skeletons

- Minimal representation of objects in an image while retaining the Euler number of the image
  - The Euler number is the number of objects in an image minus the number of holes in those objects
- As stated earlier, the skeletons of objects in an image is found by successive thinning until stability

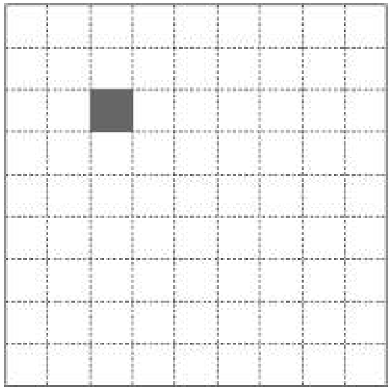




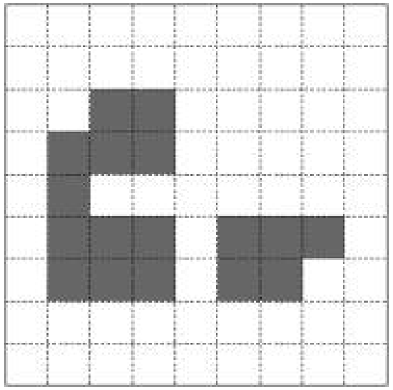
Geodesic dilation

- In geodesic dilation the result after dilating the (marker) image *f* is masked using a mask image *g*

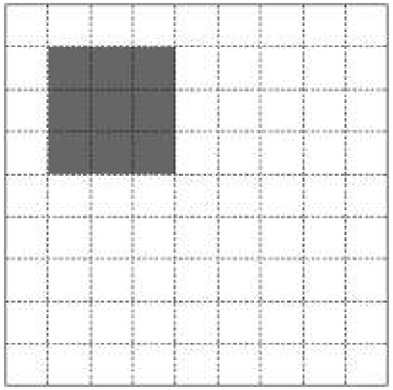
$$\delta_g(f) = \delta(f) \wedge g$$



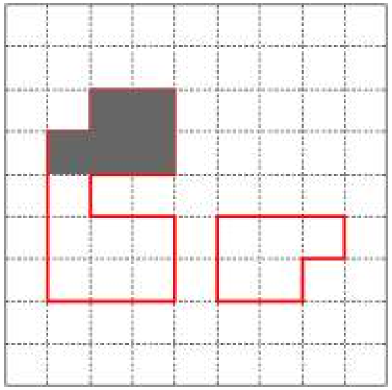
Marker



Mask



Result after dilating the marker image

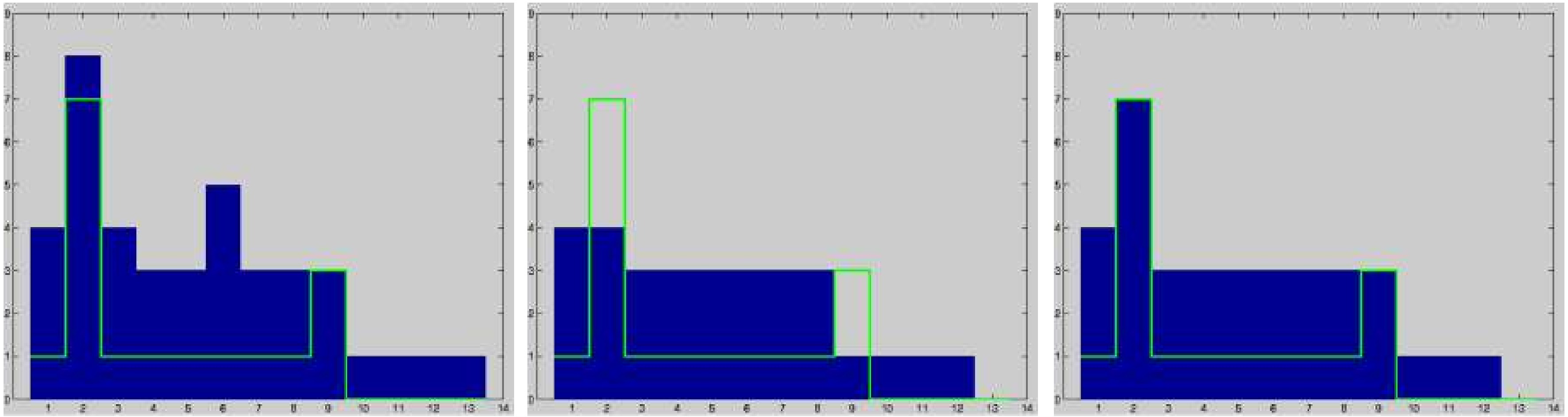


Masked result

Geodesic erosion

- Similar to geodesic dilation, the result after erosion is masked with a mask image g

$$\varepsilon_g(f) = \varepsilon(f) \vee g$$



Geodesic dilation and erosion

- Geodesic dilation repeated n times is expressed and given by

$$\delta_g^{(n)}(f) = \delta_g[\delta_g^{(n-1)}(f)]$$

- Likewise, geodesic erosion repeated n times is expressed and given by

$$\varepsilon_g^{(n)}(f) = \varepsilon_g[\varepsilon_g^{(n-1)}(f)]$$

NTNU

Reconstruction

- Reconstruction is based on geodesic dilation and erosion
- Can for instance be used to
  - Find and fill local minima or maxima
  - Suppress local minima or maxima less than a give size
  - Remove noise while not affecting structures of interest
- Reconstruction by dilation is given by:

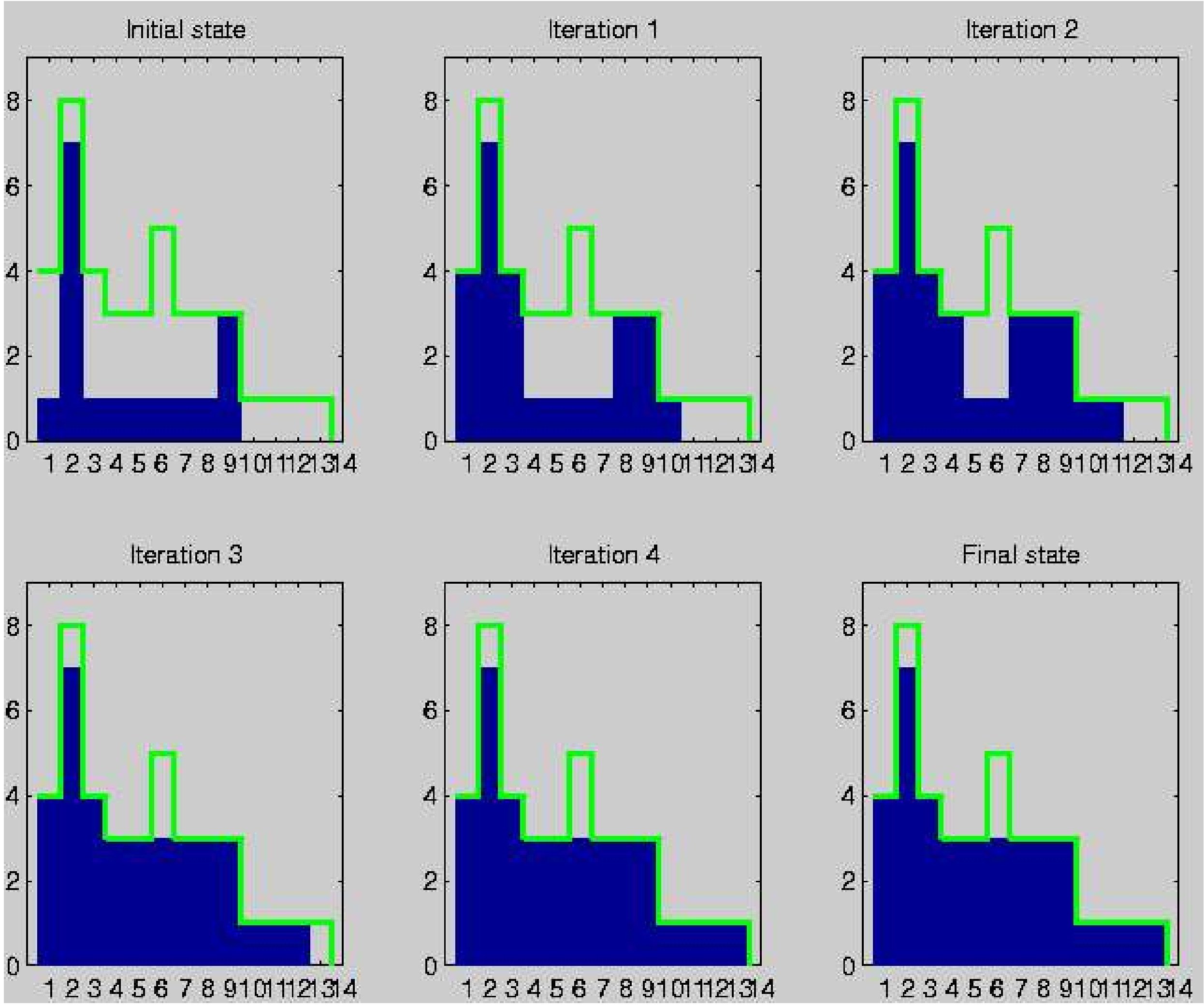
$$R_g(f) = \delta_g^{(i)}(f)$$

where

$$\delta_g^{(i)}(f) = \delta_g^{(i+1)}(f)$$

NTNU

Reconstruction by dilation, example



Reconstruction by erosion

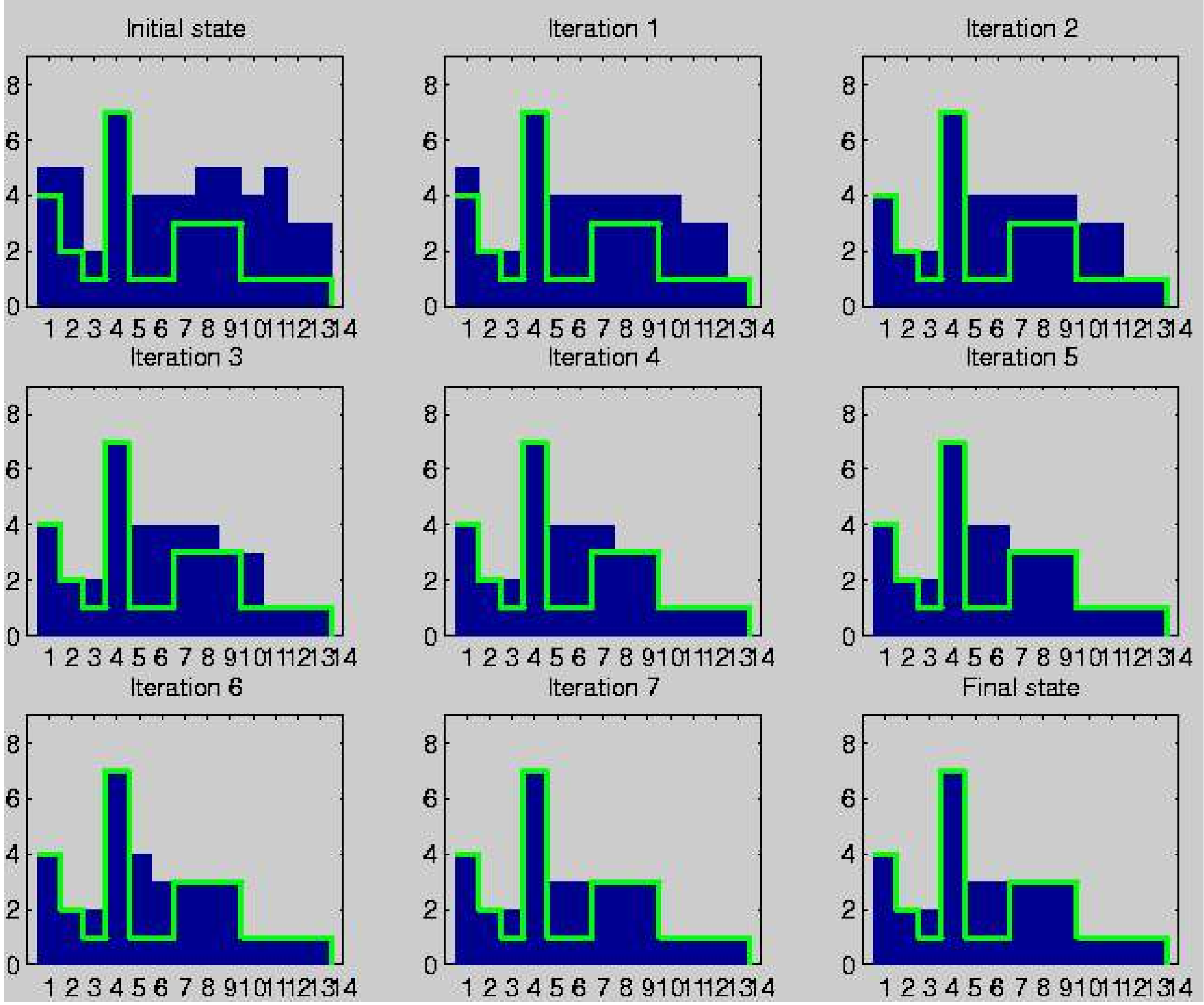
- Reconstruction by erosion is given by

$$R_g^*(f) = \varepsilon_g^{(i)}(f)$$

where

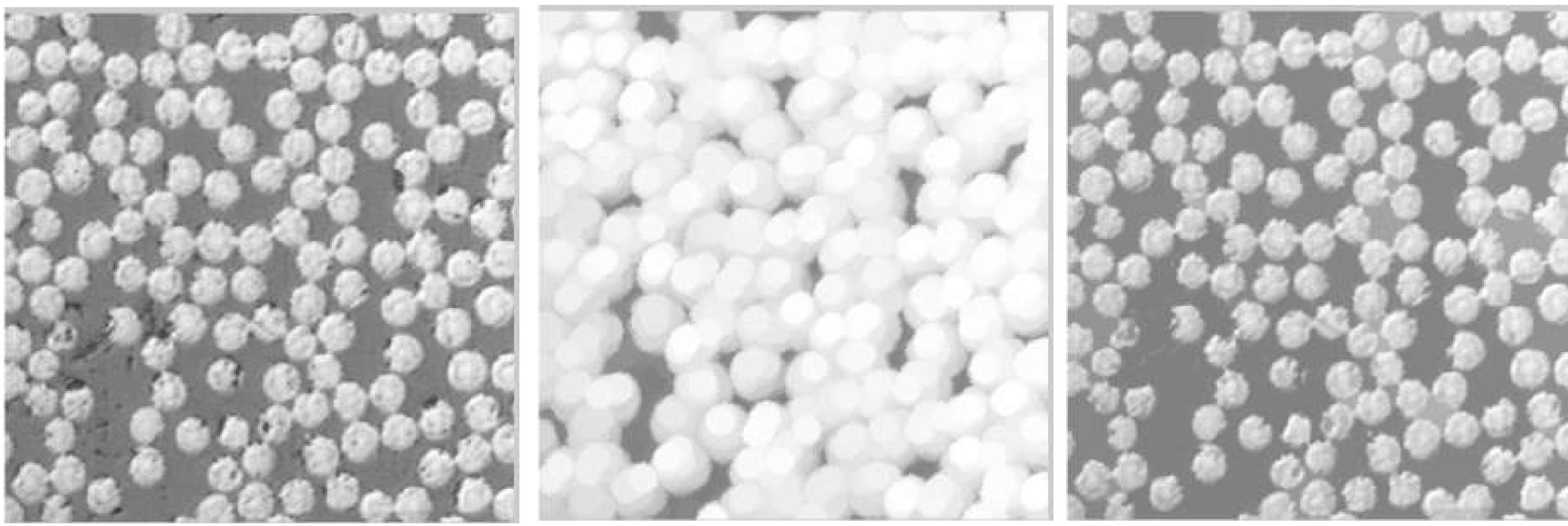
$$\varepsilon_g^{(i)}(f)\,\varepsilon_g^{(i+1)}(f)$$

Reconstruction by erosion, example



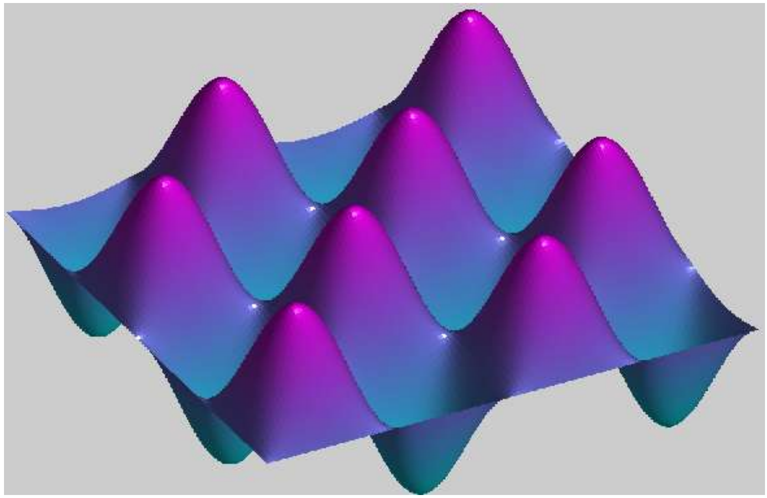
Another reconstruction example

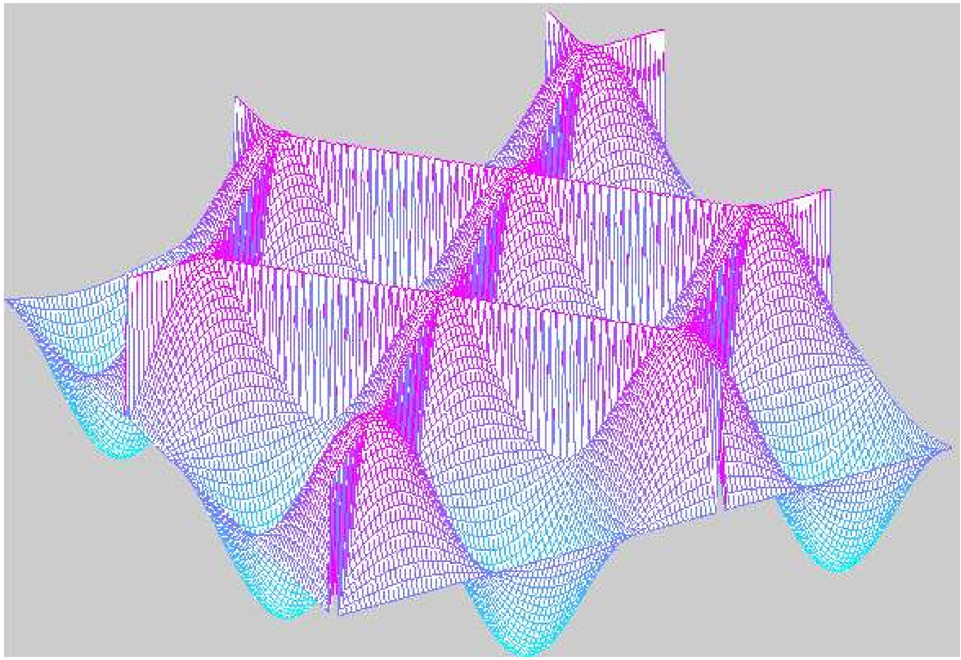
- Using reconstruction to remove noise



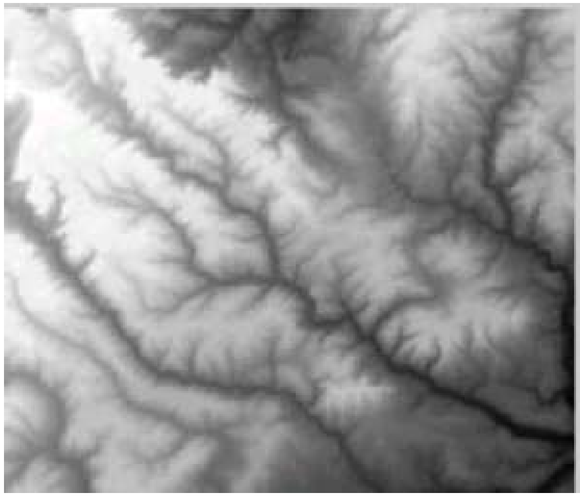
Watershed segmentation

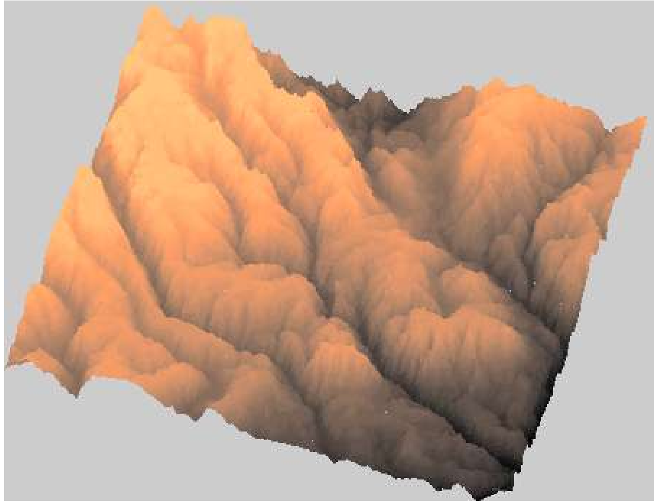
- Watershed segmentation divides an image into basins and finds the positions where the basins would meet if they were gradually filled

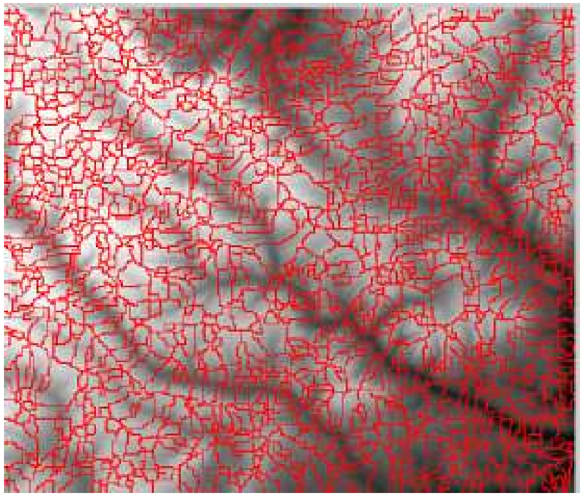




Watershed segmentation



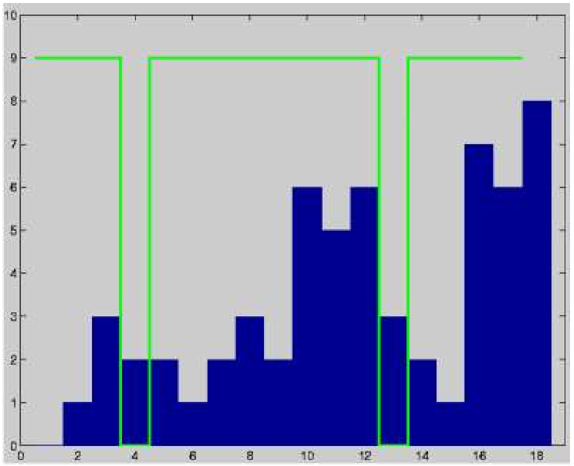


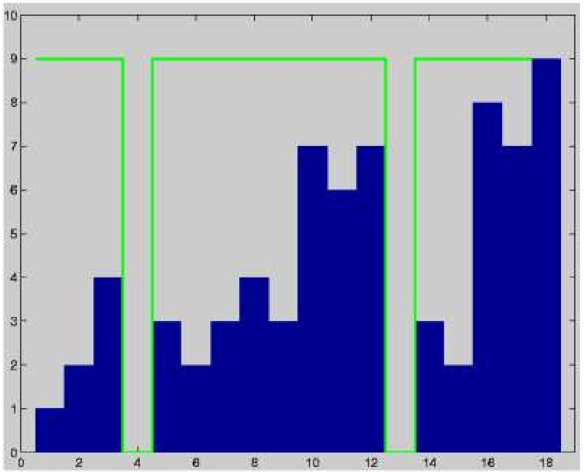


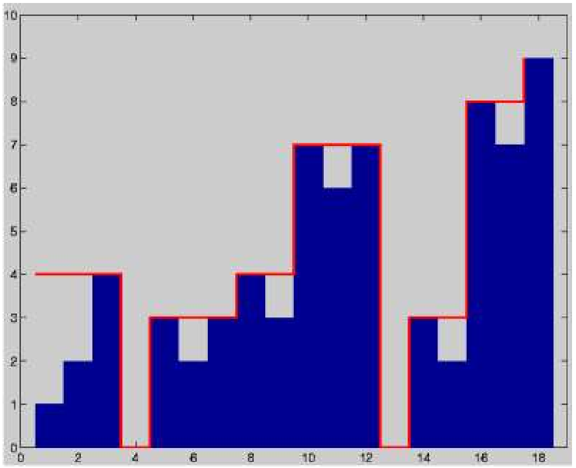
Problem: too many local minima results in oversegmentation

Watershed segmentation

- Solution: Use a marker to mark wanted minima, and fill the rest of the local minima

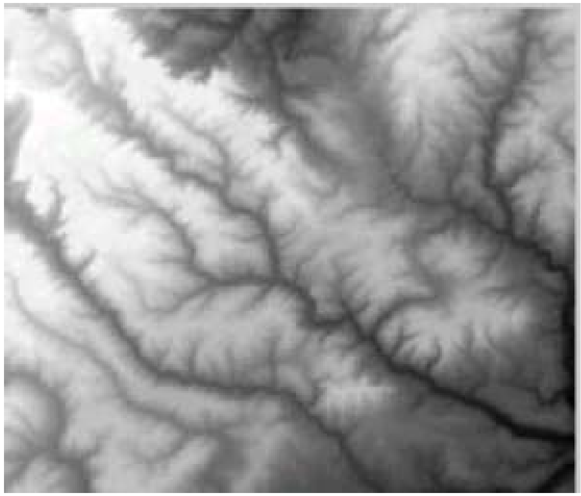


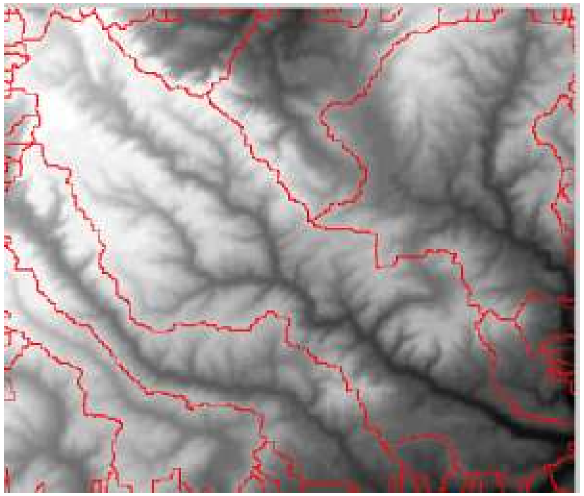




Watershed segmentation

- Final segmentation result





References

- Pierre Soille, Morphological Image Analysis. Springer-Verlag, 2003.
- Rafael C. Gonzalez, Richard E Woods. Digital Image Processing. Prentice Hall, second edition, 2002.
- Milan Sonka et. al. Image Processing, Analysis and Machine Vision. PWS Publishing, second edition, 1999.