Existence of minimizers

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be l.s.c. and let there exist an α such that the sublevelset

$$S_\alpha := \{u \in \mathbb{R}^n \mid E(u) \leq \alpha\}$$

is nonempty and bounded, then

$$\hat{u} \in \arg\min_u E(u)$$

exists.

Proof. Consider a sequence $(u_k)_k$ such that $E(u_k) \to \inf_u E(u)$. (Remember that the infimum is the largest lower bound on all possible values of $E(u)$.)

We distinguish two cases: For $\alpha = \inf_u E(u)$ the non-emptyness of S_α yields the assertion. For $\alpha > \inf_u E(u)$ it holds that from some sufficiently large k_0 on, we will have $u_k \in S_\alpha$. Since S_α is bounded there exists a convergent subsequence $u_{k_l} \to \bar{u}$. Due to the lower semi-continuity we find

$$\inf_u E(u) = \lim_{k\to\infty} E(u_k) = \lim_{l\to\infty} E(u_{k_l}) \geq E(\bar{u}).$$

Since by definition $\inf_u E(u) \leq E(\bar{u})$ we obtain equality and hence there exists $\bar{u} \in \operatorname{argmin}_u E(u)$. □

Remark: This is a fundamental strategy for showing the existence of minimizers since the arguments do not only hold in $\mathbb{R}^n$ but in arbitrary topologies after replacing "bounded" by "precompact"!

Equivalence of l.s.c. and closedness

For $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following two statements are equivalent

- E is lower semi-continuous (l.s.c.).
- E is closed.

Proof. Let E be closed and assume that E is not l.s.c.Then there exists a point u^0 and a sequence $(u_k)_k$ with $\lim_k u_k = u^0$ such that

$$\liminf_k E(u_k) < E(u^0).$$

In particular, there exists $\alpha \in \mathbb{R}$ and a subsequence $(u_{k_l})_{k_l}$ such that

$$E(u_{k_l}) \leq \alpha < E(u^0) \quad \forall k \tag{1}$$

Obviously, $(u_{k_l}, \alpha) \in \operatorname{epi}(E)$ for all k_l and $(u_{k_l}, \alpha) \to (u^0, \alpha)$, but according to (1) $(u^0, \alpha) \notin \operatorname{epi}(E)$, which contradicts the closedness of E.

Now let E be l.s.c. and assume that E is not closed. Then there exists a sequence $(u_k, \alpha_k) \in \operatorname{epi}(E)$ with $(u_k, \alpha_k) \to (u^0, \alpha^0) \notin \operatorname{epi}(E)$. We find

$$\liminf_k E(u_k) \leq \lim_k \alpha_k = \alpha^0 < E(u^0).$$

On the other hand, due to E being l.s.c. we have $E(u^0) \leq \liminf_k E(u_k)$, which is a contradiction. □

Convex functions are locally Lipschitz on int(dom(E)).

As mentioned in the lecture proving this claim in 1d is an exercise for yourself to which you find a solution below. Note that the statements holds in $\mathbb{R}^n$, too.

Proof. Part 1: Let $x, x_1, x_2 \in \operatorname{int}(\operatorname{dom}(E))$ such that $x_1 < x < x_2$. Then for $\alpha = \frac{x_2 - x}{x_2 - x_1}$ we have

$$\alpha x_1 + (1-\alpha)x_2 = \alpha(x_1 - x_2) + x_2 = x.$$

We can compute

$$\begin{aligned}
E(x) - E(x_1) &\leq \alpha E(x_1) + (1-\alpha)E(x_2) - E(x_1) \\
&= (1-\alpha)(E(x_2) - E(x_1)) \\
&= \frac{E(x_2) - E(x_1)}{x_2 - x_1}(x - x_1).
\end{aligned}$$

On the other hand

$$\begin{aligned}
E(x_2) - E(x) &\geq E(x_2) - (\alpha E(x_1) + (1-\alpha)E(x_2)) \\
&= \alpha(E(x_2) - E(x_1)) \\
&= \frac{E(x_2) - E(x_1)}{x_2 - x_1}(x_2 - x),
\end{aligned}$$

such that

$$\frac{E(x) - E(x_1)}{x - x_1} \leq \frac{E(x_2) - E(x_1)}{x_2 - x_1} \leq \frac{E(x_2) - E(x)}{x_2 - x}.$$

Now for a given $x \in \mathrm{int}(\mathrm{dom}(E))$, pick $a, b, x_1, x_2 \in \mathrm{int}(\mathrm{dom}(E))$ such that $a < x_1 < x < x_2 < b$. We claim that E is Lipschitz on $]x_1, x_2[$. For any $y_1 < y_2 \in]x_1, x_2[$ we have

$$\frac{E(y_2) - E(y_1)}{y_2 - y_1} \leq \frac{E(b) - E(y_1)}{b - y_1} \leq \frac{E(b) - E(x_2)}{b - x_2} \tag{2}$$

as well as

$$\frac{E(y_2) - E(y_1)}{y_2 - y_1} \geq \frac{E(y_2) - E(a)}{y_2 - a} \geq \frac{E(x_2) - E(a)}{x_2 - a}. \tag{3}$$

Using (2) and (3) we can conclude

$$|E(y_2) - E(y_1)| \leq \max\left(\left|\frac{E(x_2) - E(a)}{x_2 - a}\right|, \left|\frac{E(b) - E(x_2)}{b - x_2}\right|\right) |y_2 - y_1|,$$

such that E is Lipschitz on $]x_1, x_2[$. □

Example of a convex function that is not continuous.

$$E(u) = \begin{cases} u & \text{if } u > 0 \\ 1 & \text{if } u = 0 \\ \infty & \text{else.} \end{cases}$$

The subdifferential $\partial E(u)$ is nonempty and bounded for $u \in \mathrm{int}(\mathrm{dom}(E))$

We will use

Theorem 1 (Supporting Hyperplane Theorem). *Let $S \subset \mathbb{R}^{n+1}$ be a convex set and let $z \in \partial S$. Then there exists a supporting hyperplane of S which contains z.*

Proof. Step 1: Show that $\partial E(u)$ is nonempty for $u \in \mathrm{ri}(\mathrm{dom}(E))$:
The point $(u, E(u))$ is on the boundary of $\mathrm{epi}(E)$. Thus, by the supporting hyperplane theorem there exist $0 \neq (q, r) \in \mathbb{R}^{n+1}$, $b \in \mathbb{R}$ such that

$$\left\langle \begin{bmatrix} q \\ r \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} \right\rangle \leq b \quad \forall (v, \alpha) \in \mathrm{epi}(E),$$

and $b = \left\langle \begin{bmatrix} q \\ r \end{bmatrix}, \begin{bmatrix} u \\ E(u) \end{bmatrix} \right\rangle$. In other words,

$$\left\langle \begin{bmatrix} q \\ r \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} - \begin{bmatrix} u \\ E(u) \end{bmatrix} \right\rangle \leq 0 \quad \forall (v, \alpha) \in \mathrm{epi}(E).$$

We have to exclude a vertical hyperplane, i.e. $r = 0$. Assume $r = 0$. Then

$$\langle q, v - u \rangle \leq 0, \qquad \forall v \in \mathrm{dom}(E).$$

Since $u \in \mathrm{int}(\mathrm{dom}(E))$ there exists an $\epsilon > 0$ such that $u + \epsilon q \in \mathrm{dom}(E)$ which means

$$\epsilon \|q\|^2 \leq 0 \quad \Rightarrow \quad q = 0,$$

and contradicts $0 \neq (q, r)$.

To get the "right" inequality, we have to make sure $r < 0$. Assume $r > 0$. Then $v = u$ and $\alpha > E(u)$ violates the supporting hyperplane inequality.

Thus $r < 0$ and we find

$$\left\langle \begin{bmatrix} q/r \\ 1 \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} - \begin{bmatrix} u \\ E(u) \end{bmatrix} \right\rangle \geq 0 \quad \forall (v, \alpha) \in \mathrm{epi}(E),$$

which in particular means

$$E(v) - E(u) + \langle q/r, v - u \rangle \geq 0, \qquad \forall v \in \mathrm{dom}(E),$$

or $-q/r \in \partial E(u)$.

Step 2: Show that $\partial E(u)$ is bounded for $u \in \mathrm{int}(\mathrm{dom}(E))$. For $\epsilon > 0$ sufficiently small, we have

$$M := \bigcup_{k \in \{1, \cdots, n\}} \{u + \epsilon e_k,\ u - \epsilon e_k\} \subset \mathrm{dom}(E).$$

Now let $b := \max_{q \in M} E(q)$, and for every $p \in \partial E(u)$ note that there exists a point $q \in M$ with

$$E(q) - E(u) - \underbrace{\langle p, q - u \rangle}_{= \epsilon \|p\|_\infty} \geq 0.$$

Therefore,

$$\frac{b - E(u)}{\epsilon} \geq \frac{E(q) - E(u)}{\epsilon} \geq \|p\|_\infty,$$

is a bound on the infinity norm of all elements in $\partial E(u)$. □

For a convex function E that is differentiable at $u \in \mathrm{int}(\mathrm{dom}(E))$ it holds that $\partial E(u) = \{\nabla E(u)\}$

Proof. The subdifferential $\partial E(u)$ of some convex E at $u \in \mathrm{dom}(f)$ is given as

$$\{p \in \mathbb{R}^n : E(z) - E(u) - \langle p, z - u \rangle \geq 0, \forall\, z \in \mathrm{dom}(f)\}.$$

Since $u \in \mathrm{int}(\mathrm{dom}(E))$, we find that for all $v \in \mathbb{R}^n$, $z = u \pm \epsilon v \in \mathrm{dom}(E)$ for ϵ small enough. Therefore, it holds that

$$E(u + \epsilon v) \geq E(u) + \epsilon \langle p, v \rangle, \quad E(u - \epsilon v) \geq E(u) - \epsilon \langle p, v \rangle,$$

for all $v \in \mathbb{R}^n$ and ϵ small enough. This implies that

$$\lim_{\epsilon \to 0} \frac{E(u + \epsilon v) - E(u)}{\epsilon} \geq \langle p, v \rangle, \quad \lim_{\epsilon \to 0} \frac{E(u) - E(u - \epsilon v)}{\epsilon} \leq \langle p, v \rangle,$$

which means (using the hint)

$$\langle \nabla E(u), v\rangle \geq \langle p, v\rangle, \quad \langle \nabla E(u), v\rangle \leq \langle p, v\rangle,$$

i.e.

$$\langle \nabla E(u) - p, v\rangle = 0$$

for all $v \in \mathbb{R}^n$. For the particular choice of $v := \nabla E(u) - p$ we find $p = \nabla f(u)$.

The above concludes the proof if we can show that $\partial f(u)$ is non-empty, which follows from the Theorem "Subdifferentiability" in the lecture. $\square$