Convergence of Krasnosel'skii-Mann Iteration

We'll make use of the identity

$$\|(1-\theta)a+\theta b\|^2 = (1-\theta)\|a\|^2 + \theta\|b\|^2 - \theta(1-\theta)\|a-b\|^2,$$

which holds for any $\theta \in \mathbb{R}$, $a, b \in \mathbb{R}^n$ as we have seen in the exercises.

Because G is averaged, there exists a non-expansive mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $G = (1-\theta)I + \theta T$. Note that T has the same fixed points as F:

$$u^* = Tu^* \;\Leftrightarrow\; (1-\theta)u^* + \theta u^* = (1-\theta)u^* + \theta Tu^* \;\Leftrightarrow\; u^* = [(1-\theta)I + \theta T]\, u^* = Gu^* \tag{1}$$

We consider the fixed point iteration

$$u^{k+1} = G(u^k) = (1-\theta)u^k + \theta Tu^k.$$

Denote by U the (nonempty) set of fixed-points of G and let $u^* \in U$, i.e. $G(u^*) = u*$. Then we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &= \|(1-\theta)(u^k - u^*) + \theta(Tu^k - u^*)\|^2 \\
&= (1-\theta)\|u^k - u^*\|^2 + \theta\|Tu^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \\
&= (1-\theta)\|u^k - u^*\|^2 + \theta\|Tu^k - Tu^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \qquad (*) \\
&\le (1-\theta)\|u^k - u^*\|^2 + \theta\|u^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2 \\
&= \|u^k - u^*\|^2 - \theta(1-\theta)\|Tu^k - u^k\|^2
\end{aligned}$$

This shows that the so called Fejèr monotonicity of the fixed point iteration, i.e., the distance to the solution set decreases at each step.

Applying the inequality k times yields

$$\|u^{k+1} - u^*\|^2 \le \|u^0 - u^*\|^2 - \theta(1-\theta)\sum_{j=0}^{k}\|Tu^j - u^j\|^2$$

and hence

$$\sum_{j=0}^{k}\|Tu^j - u^j\|^2 \le \frac{\|u^0 - u^*\|^2}{\theta(1-\theta)},$$

which implies that $\|Tu^k - u^k\| \to 0$, for $k \to \infty$.

From that we can also estimate a convergence rate of the fixed-point residual:

$$\min_{j=0\ldots k}\|Tu^j - u^j\|^2 \le \frac{\|u^0 - u^*\|^2}{(k+1)\theta(1-\theta)},$$

Since the iterates $\{u^k\}_{k=1}^{\infty}$ lie in the compact set (due to the Fejèr monotonicity)

$$\{u^k\}_{k=1}^{\infty} \subset C = \left\{v \mid \|v - u^*\| \le \|u^0 - u^*\|\right\},$$

there exists at least one subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ which converges to some point $\widehat{u}$.

Since $Tu^{k_l} - u^{k_l} \to 0$, we also have that $Gu^{k_l} - u^{k_l} = (G-I)u^{k_l} \to 0$. Since $G-I$ is Lipschitz continuous (as T is nonexpansive) and hence continuous, we have that $G\widehat{u} = \widehat{u}$ and hence the subsequence converges to a point in $\widehat{u} \in U$.

As (*) holds for any point from $u^* \in U$, we can apply it the point $\widehat{u}$ our subsequence converges to. We know that for the iterates of the original sequence the distance to this point is monotonically decreasing,

$$\|u^{k+1} - \widehat{u}\| \le \|u^k - \widehat{u}\|.$$

Since a subsequence $\{u^{k_l}\}_{l=1}^{\infty}$ of $\{u^k\}_{k=1}^{\infty}$ is converging to $\widehat{u}$, and $\|u^k - \widehat{u}\|$ is monotonically decreasing, we have convergence of the entire sequence to $\widehat{u}$.