A continuously differentiable convex function $E : \mathbb{R}^n \to \mathbb{R}$ is L-smooth if and only if $\frac{1}{L}\nabla E$ is firmly nonexpansive

If $\frac{1}{L}\nabla E$ is firmly nonexpansive, the L-smoothness is a simple conclusion of the Cauchy-Schwarz inequality.

Now let $E : \mathbb{R}^n \to \mathbb{R}$ be L-smooth. We show the assertion in two steps

1. L-smoothness implies that $\frac{L}{2}\|u\|^2 - E(u)$ is convex.
2. If $\frac{L}{2}\|u\|^2 - E(u)$ is convex, then $\frac{1}{L}\nabla E$ is firmly nonexpansive.

The first step becomes easy if we assume we know that for a differentiable function E convexity is equivalent to monotononicity of the gradient, i.e.

$$\langle \nabla E(u) - \nabla E(v), u - v \rangle \geq 0.$$

In this case, L-smoothness quickly yields the convexity of $\frac{L}{2}\|u\|^2 - E(u)$.

Let a function $G(u) = \frac{L}{2}\|u\|^2 - E(u)$ be convex and differentiable, and fix some point u. Then it holds that all v that

$$\begin{aligned}
0 \leq & G(v) - G(u) - \langle \nabla G(u), v - u \rangle \\
&= \frac{L}{2}\|v\|^2 - E(v) - \frac{L}{2}\|u\|^2 + E(u) - \langle Lu - \nabla E(u), v - u \rangle \\
&= E(u) - E(v) + \frac{L}{2}\|v\|^2 - \frac{L}{2}\|u\|^2 + L\|u\|^2 - L\langle u, v \rangle - \langle \nabla E(u), v - u \rangle \\
&= E(u) - E(v) + \frac{L}{2}\|v\|^2 - L\langle u, v \rangle + \frac{L}{2}\|u\|^2 - \langle \nabla E(u), v - u \rangle \\
&= E(u) - E(v) + \frac{L}{2}\|v - u\|^2 - \langle \nabla E(u), v - u \rangle
\end{aligned}$$

In other words, we derived the quadratic upper bound

$$\min_z E(z) \leq E(v) \leq E(u) - \langle \nabla E(u), v - u \rangle + \frac{L}{2}\|v - u\|^2.$$

Minimizing the right hand side yields $v^* = u - \frac{1}{L}E(u)$ and therefore

$$\min_z E(z) \leq E(u) - \frac{1}{2L}\|\nabla E(u)\|^2. \tag{1}$$

Now consider the function

$$E_v(w) = E(w) - \langle \nabla E(v), w \rangle.$$

It holds that E_v is L-smooth and convex, since it is just a linear perturbation of E. Furthermore, $\nabla E_v(v) = 0$, such that v has to minimize E_v. According to (1) it holds for any element u that

$$\begin{aligned}
E_v(v) &\leq E_v(u) - \frac{1}{2L}\|\nabla E_v(u)\|^2 \\
&= E_v(u) - \frac{1}{2L}\|\nabla E(u) - \nabla E(v)\|^2
\end{aligned}$$

But symmetry also

$$\begin{aligned} E_u(u) &\leq E_u(v) - \frac{1}{2L}\|\nabla E_u(v)\|^2 \\ &= E_u(v) - \frac{1}{2L}\|\nabla E(v) - \nabla E(u)\|^2. \end{aligned}$$

Adding these two inequalities one obtains

$$E_v(v) + E_u(u) \leq E_u(v) + E_v(u) - \frac{1}{L}\|\nabla E(u) - \nabla E(v)\|^2$$

and we compute

$$\begin{aligned} &E_v(v) + E_u(u) - E_u(v) - E_v(u) \\ =&E(v) - \langle \nabla E(v), v\rangle + E(u) - \langle \nabla E(u), u\rangle - E(v) + \langle \nabla E(u), v\rangle - E(u) + \langle \nabla E(v), u\rangle \\ =&\langle \nabla E(v), u - v\rangle - \langle \nabla E(u), u - v\rangle \\ =&\langle \nabla E(v) - E(u), u - v\rangle \end{aligned}$$

which yields the assertion.