Chapter 1
Basics and necessary tools

Variational Methods for Computer Vision
WS 17/18

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Math basics

Signals, images, representations

Variational methods
- An example
- Understanding ill-posedness
- Optimality conditions
  - Discrete case
  - Continuous case

Optimization
- Linear systems
- Images are discontinuous
- The gradient descent algorithm

Repeating some math basics

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Notation, norm, inner product

We will mostly work in the vector space $\mathbb{R}^n$ equipped with an inner product

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i$$

for $x, y \in \mathbb{R}^n$.

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The ℓ^2 norm is *induced* by this inner product, i.e.

$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

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There are other norms, e.g. the ℓ^1 or the ℓ^∞ norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad , \qquad \|x\|_\infty = \max_i |x_i|$$

which we will use less frequently.

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More norms

One can also define norms on function spaces! E.g. for a function $f : [0, 1] \to \mathbb{R}$ we define

$$\|f\|_2 = \sqrt{\int_0^1 (f(x))^2 \, dx}.$$

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As we will discuss in a couple of slides, one can also integrate in multiple variables and define norms on functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$, e.g.

$$\|f\|_2 = \sqrt{\int_U (f(x))^2 \, dx}.$$

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Continuity

We will need the concept of *continuous* functions. Do you remember when

$$f : \mathbb{R} \to \mathbb{R}$$

is continuous?

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The previous definition generalizes to vector valued functions!

f is continuous at x_0 if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all x with $\|x - x_0\| \leq \delta$ it holds that $\|f(x) - f(x_0)\| \leq \epsilon$.

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Mutlivariate derivatives

We will need to take derivatives of functions

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Do you remember/know how?

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Do you remember/know how?

Definition: Jacobi matrix

For a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with continuous partial derivatives we write $f \in C^1$ and call

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & . & . & \frac{\partial f_1}{\partial x_n}(x) \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_m}{\partial x_1}(x) & . & . & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^{m \times n}$$

the *Jacobi matrix* of f at $x \in U$. It is the first derivative of multivariate functions. Jf itself is a continuous function $Jf : U \to \mathbb{R}^{m \times n}$.

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Example on the board $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x - y\|^2$.

Composite functions

Chain rule for multivariate functions

Let

$$f : \mathbb{R}^n \to \mathbb{R}^m \in C^1 \qquad \text{and} \qquad g : \mathbb{R}^m \to \mathbb{R}^k \in C^1.$$

Then the composite function $(g \circ f) : \mathbb{R}^n \to \mathbb{R}^k$ is continuously differentiable and its Jacobian $J(g \circ f)$ is given by

$$J(g \circ f)(x) = (Jg)(f(x)) \cdot Jf(x).$$

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Example on the board $g:\mathbb{R}^m \to \mathbb{R}$, $g(x) = \frac{1}{2}\|x-y\|^2$, $f:\mathbb{R}^n \to \mathbb{R}^m$, $f(z) = Az$ for some matrix $A \in \mathbb{R}^{m\times n}$.

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Multivariate integration

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$$f : U \subset \mathbb{R}^n \to \mathbb{R}$$

Do you remember/now how?

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Integrate with respect to all variables sequentially!

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Example:

$$f : \{(x, y) \mid 1 \leq x \leq 4, \ -2 \leq y \leq 1\} \to \mathbb{R}$$
$$f(x, y) = y^2 + 1 + \sin(x)y$$

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Another example:

$$f : \{(x, y) \mid x^2 + y^2 \leq 1\} \to \mathbb{R}$$
$$f(x, y) = 1 + x\cos(y^6)$$

Eigendecompositions

Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$ be a matrix. We say $\lambda \in \mathbb{R}$ is an *eigenvalue* of A if there exists a $v \neq 0$ such that

$$Av = \lambda v.$$

The corresponding v is called an *eigenvector*.

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If there exist matrices $U \in \mathbb{R}^{n\times n}$ with $U^T U = UU^T = I$, and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that

$$A = UDU^T$$

we call this an *eigendecomposition* of A. The diagonal elements of D are eigenvalues of A.

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Singular value decomposition

Not every matrix $A \in \mathbb{R}^{n \times n}$ has a diagonal eigendecomposition over $\mathbb{R}$. It often is useful to use a *singular value decomposition*, which even works for matrices $A \in \mathbb{R}^{n \times m}$.

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Singular value decomposition (SVD)

For any $A \in \mathbb{R}^{n \times m}$ there exist orthogonal matrices $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{m \times m}$, and a non-negative diagonal matrix $D \in \mathbb{R}^{n \times m}$ such that

$$A = UDV^T$$

The diagonal entries of D are called singular values.

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The diagonal entries of D are called singular values.

The number of nonzero singular values is the *rank* of A.

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Signal Representation

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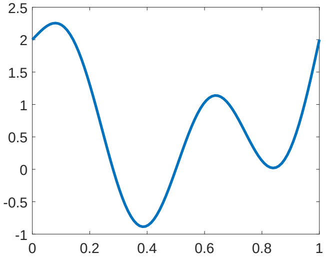
How do we represent signals?



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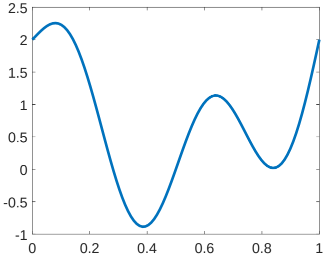

Continuous: Functions

$$
\begin{aligned}
f : [a, b] &\to \mathbb{R} \\
x &\mapsto f(x)
\end{aligned}
$$

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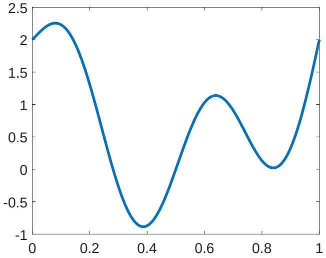

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Discrete: Vectors $f \in \mathbb{R}^n$

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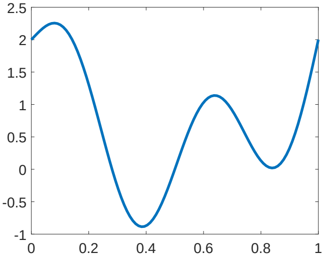

Continuous: Functions

$$\begin{aligned} f : [a,b] &\to \mathbb{R} \\ x &\mapsto f(x) \end{aligned}$$

Discrete: Vectors $f \in \mathbb{R}^n$

One typically interprets/relates:

$$f_i = f(x_i), \qquad x_i = a + (i-1) \cdot \frac{b-a}{n-1}, \qquad \text{for } i \in \{1, \cdots, n\}.$$

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How do we represent images?

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How do we represent images?

Continuous: Functions

Grayscale

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto f(x)$$

Color

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$
$$x \mapsto f(x) = (f_R(x), f_G(x), f_B(x))^T$$

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Discrete: Matrices and Tensors

Grayscale

$f \in \mathbb{R}^{n \times m}$

Color

$f \in \mathbb{R}^{n \times m \times 3}$

The points $x_{i,j}$ at which the continuous function f is sampled to obtain its discrete representation are called *pixels*.

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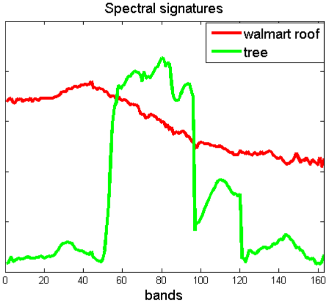
Many more types of image data

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n \qquad \text{or} \qquad f : (\Omega \times \Gamma) \subset \mathbb{R}^3 \to \mathbb{R}$$

E.g. hyperspectral images.

Hyperspectral cube with 163 bands



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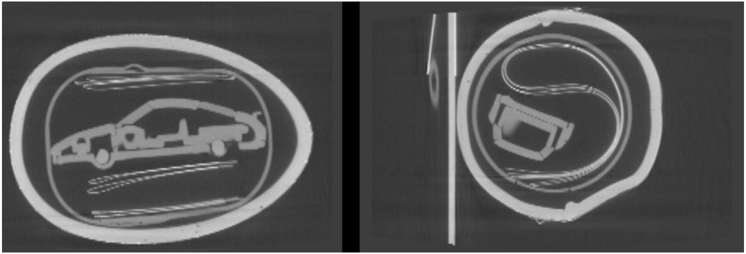
E.g. medical imaging - three spatial dimension.



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Many more types of image data

$f : (\Omega \times \Gamma) \subset \mathbb{R}^3 \to \mathbb{R}^3$

E.g. color videos.

▸ Coffee?

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More types of discretization

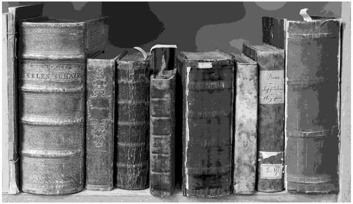
Besides the discretization of the *domain* Ω

$$f : \Omega \to \mathbb{R} \qquad \to \qquad f : \{x_{1,1}, \cdots, x_{n,m}\} \to \mathbb{R}$$

digital images may also have a discrete *range*, e.g.,

$$f : \{x_{1,1}, \cdots, x_{n,m}\} \to \{0, \cdots, 255\}$$

for an $8-$ *bit* quantization.



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Variational methods

Define an energy E on continuous images, i.e.,

$$E : \mathcal{X} \to \mathbb{R} \cup \{\infty\} \tag{1}$$

from a suitable space $\mathcal{X}$ of images (typically a Banach space) to the extended real numbers, such that

- u with desirable properties $\to E(u)$ small,
- unrealistic/"bad" $u \to E(u)$ large.

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If $\mathcal{X}$ is a function space (continuous formulation of images), then E is a function that maps functions to real numbers. We call E a *functional*.

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For $\mathcal{X}$ being a function space, determining the solution of an imaging problem by determining

$$\hat{u} = \operatorname*{argmin}_u E(u),$$

is called a *variational method*.

Calculus of variation

Analyzing variational methods

$$\hat{u} = \operatorname*{argmin}_u E(u),$$

in terms of existence, uniqueness, optimality conditions and properties of the solution can be mathematically challenging and requires *functional analysis*.

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We will

- Often formulate energies in a continuous setting.

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- Not require prior knowledge in functional analysis.

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$$\hat{u} = \underset{u}{\operatorname{argmin}}\, E(u),$$

in terms of existence, uniqueness, optimality conditions and properties of the solution can be mathematically challenging and requires *functional analysis*.

We will

- Often formulate energies in a continuous setting.
- Not require prior knowledge in functional analysis.
- Occasionally do some analysis in infinite dimensions/function spaces.

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Calculus of variation

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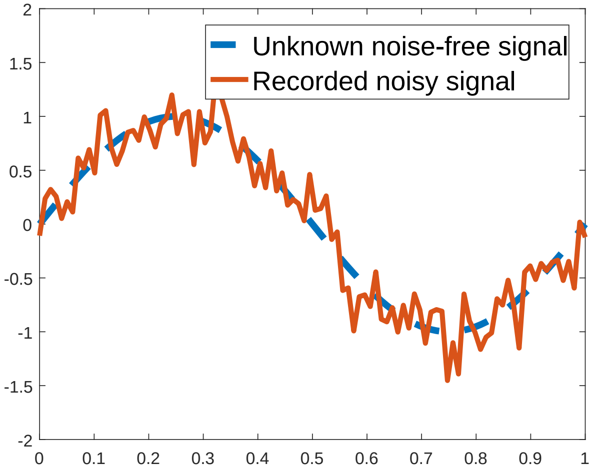
We will

- Often formulate energies in a continuous setting.
- Not require prior knowledge in functional analysis.
- Occasionally do some analysis in infinite dimensions/function spaces.
- Often turn to a discrete point of view and use analysis instead of functional analysis.

A simple example

Let us consider a simple example:



How can we reduce the noise?

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A simple example

The denoised signal should still look somewhat similar to the input data. But how should we measure similarity? Simple choice:

$$H_f(u) = \int_0^1 (u(x) - f(x))^2 \, dx =: \|u - f\|_2^2.$$

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The denoised signal should be smoother, i.e., contain less oscillations. We need a regularization R that penalizes rapid changes of the signal! Simple choice:

$$R(u) = \int_0^1 \left(\partial_x u(x)\right)^2 \, dx = \|\partial_x u\|_2^2.$$

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Overall variational method:

$$\hat{u} = \operatorname*{argmin}_u H_f(u) + \alpha R(u).$$

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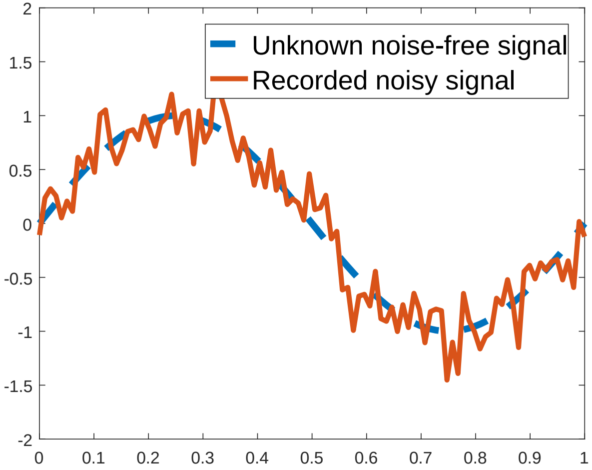
A simple example

Result of

$$\hat{u} = \underset{u}{\operatorname{argmin}} \|u - f\|_2^2 + 10 \cdot \|\partial_x u\|_2^2$$



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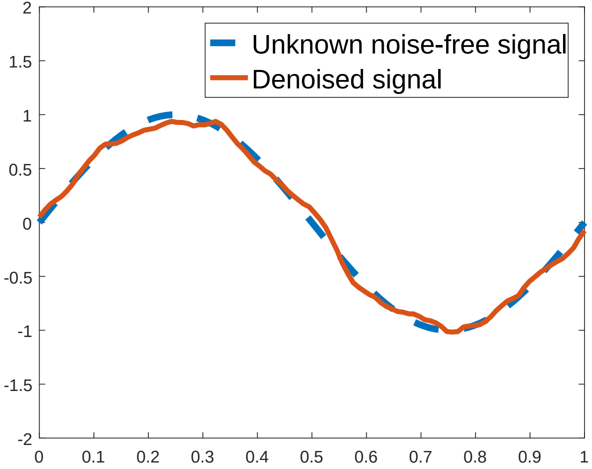

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A simple example

For the computation I, of course, discretized

$$\hat{u} = \operatorname*{argmin}_{u} \|u - f\|_2^2 + 10 \cdot \|\partial_x u\|_2^2$$

and used

$$\begin{aligned}\mathbb{R}^n \ni \hat{u} &= \operatorname*{argmin}_{u \in \mathbb{R}^n} \sum_{i=1}^{n} (u_i - f_i)^2 + 10 \cdot \sum_{i=2}^{n} (u_i - u_{i-1})^2, \\ &= \operatorname*{argmin}_{u \in \mathbb{R}^n} \|u - f\|_2^2 + 10 \cdot \|Du\|_2^2,\end{aligned}$$

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with the discrete derivative matrix

$$\mathbb{R}^{n-1 \times n} \ni D = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

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Why care about a continuous representation?

For our simple example, we had two formulations:

Continuous:

$$\hat{u} = \underset{u}{\operatorname{argmin}} \int_0^1 (u(x) - f(x))^2 \, dx + \alpha \int_0^1 \left(\partial_x u(x)\right)^2 \, dx$$

Discrete:

$$\hat{u} = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \sum_{i=1}^{n} (u_i - f_i)^2 + \alpha \cdot \sum_{i=2}^{n} (u_i - u_{i-1})^2$$

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Discrete:

$$\hat{u} = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \sum_{i=1}^{n} (u_i - f_i)^2 + \alpha \cdot \sum_{i=2}^{n} (u_i - u_{i-1})^2$$

Why should we care about a continuous formulation at all, if the computer can only compute discrete solutions anyways?

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Reasons for variational methods (continuous formulation)

1. Beautifully concise formulation.

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Reasons for variational methods (continuous formulation)

1. Beautifully concise formulation.

2. Independence of the discretization.

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Reasons for variational methods (continuous formulation)

1. Beautifully concise formulation.

2. Independence of the discretization.

3. Some effects can only be explained in a continuous setting!

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Differentiation

Data from: *Microsoft Research GeoLife GPS Trajectories*

Time	'12:44:12'	'12:44:13'	'12:44:15'
Latitude	39.974408918	39.974397078	39.973982524
Longitude	116.30352210	116.30352693	116.30362184

How fast did this person go?

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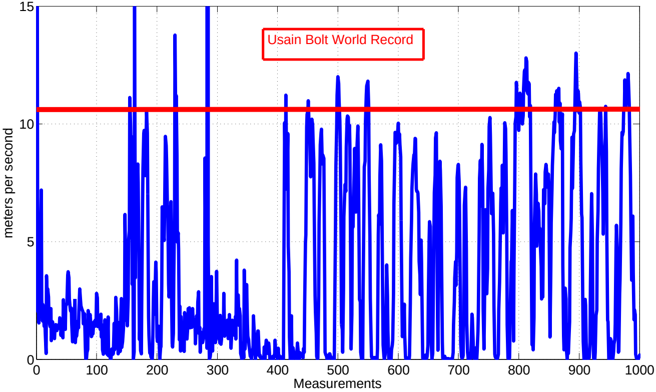
Differentiation



New world record? Top speed of 161.78 km/h?

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What went wrong?

Something makes the problem of differentiation nasty...

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What went wrong?

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Definition (Well-posed problems (Hadamard))

A problem is *well-posed* if the following three properties hold.

1. **Existence**: For all suitable data, a solution exists.
2. **Uniqueness**: For all suitable data, the solution is unique.
3. **Stability**: The solution depends continuously on the data.

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1. **Existence**: For all suitable data, a solution exists.
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3. **Stability**: The solution depends continuously on the data.

Definition (Ill-posed problems)

A problem that violates any of the three properties of well-posedness is called an *ill-posed problem*.

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Stability?

What does stability really mean?

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Stability?

What does stability really mean?

Continuous dependence on the data

Let f^δ be the measured data, and $I(f,\delta)$ the operation of recovering our desired solution (assuming existence and uniqueness).

We say that the solution depends continuously on the data if for any $f^\delta = f + n^\delta$ with $\|n^\delta\| \leq \delta$ it holds that $\|I(f,0) - I(f^\delta,\delta)\| \to 0$ as $\delta \to 0$. In other words, I is continuous.

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Differentiation is ill-posed

Computation on the board:

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Differentiation is ill-posed

Computation on the board:

Ill-posedness of differentiation

For $f, f^\delta \in C^1([0,1])$, although the error in the data

$$\|f - f^\delta\| \leq \delta$$

is arbitrary small, the error between the derivatives

$$\|\partial_x f - \partial_x f^\delta\|$$

can be arbitrary large!

We understood the behavior in the continuous setting, i.e. independent of the discretization. What can we do?

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$\rightarrow$ **Variational Methods!**

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Variational methods can fight ill-posedness

The intuitive statement

Variational methods behave nicely

Variational methods allow to re-establish the continuous dependence on the data and therefore stabilize the problem!

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Variational methods behave nicely

Variational methods allow to re-establish the continuous dependence on the data and therefore stabilize the problem!

Exemplary mathematical result

Proposition

Let $f \in C_0^2([0,1])$ be twice continuously differentiable. If we determine

$$u^\alpha = \underset{u}{\operatorname{argmin}} \|u - f^\delta\|^2 + \alpha \cdot \|\partial_x u\|^2,$$

subject to $u(0) = u(1) = 0$, then there is a parameter choice rule $\alpha = \alpha(\delta)$ such that

$$\|\partial_x u^\alpha - \partial_x f\| \overset{\delta \to 0}{\to} 0.$$

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Why not discrete?

Discrete (=finite dimensional) linear inverse problems never violate the criterion of ”continuous dependence on the data”!

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Why not discrete?

Discrete (=finite dimensional) linear inverse problems never violate the criterion of "continuous dependence on the data"!

Let

$$D = \frac{1}{h} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

be the finite difference matrix. Then

$$\lim_{\delta \to 0} Df^\delta = D\left(\lim_{\delta \to 0} f^\delta\right) = Df,$$

since matrices are continuous operators. This holds for any matrix D.

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since matrices are continuous operators. This holds for any matrix D.

$\rightarrow$ **The discrete problem is not ill-posed!**

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An imaging example for ill-posedness

Original image

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An imaging example for ill-posedness

Blurry image $f = k * u$

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An imaging example for ill-posedness

Reconstructed image $u = \mathcal{F}^{-1}(\mathcal{F}(f)/\mathcal{F}(k))$

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An imaging example for ill-posedness

Blurry noisy image $f = k * u + n, \Rightarrow \mathcal{F}(f) \approx \mathcal{F}(k) \cdot \mathcal{F}(u)$

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An imaging example for ill-posedness

Reconstruction by $\mathcal{F}^{-1}(\mathcal{F}(f)/\mathcal{F}(k))$

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Summary

Variational methods

$$\hat{u} = \operatorname*{argmin}_{u} \underbrace{H_f(u)}_{\text{data term}} + \underbrace{\alpha}_{\substack{\text{regularization}\\ \text{parameter}}} \underbrace{R(u)}_{\text{regularization}}$$

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Many practical problems do not depend on the data continuously, they are ill-posed!

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Many practical problems do not depend on the data continuously, they are ill-posed!

Seen for the example of taking the derivative: Vartiational methods can stabilize such problems.

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Besides a more concise formulation, the effect of ill-posedness could only be explained in function spaces.

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$\rightarrow$ **Let us investigate variational methods in more detail! What are optimality conditions?**

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Let us start with the simple (discrete) case:

$$\hat{u} = \underset{u \in \mathbb{R}^n}{\text{argmin}} \sum_{i=1}^{n} (u_i - f_i)^2 + \alpha \cdot \sum_{i=2}^{n} (u_i - u_{i-1})^2$$

What is a necessary condition for optimality?

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What is a necessary condition for optimality?

The gradient with respect to u is zero, i.e.,

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What is a necessary condition for optimality?

The gradient with respect to u is zero, i.e.,

$$\begin{aligned} 0 &= 2(u_i - f_i) + 2\alpha(u_i - u_{i-1}) + 2\alpha(u_i - u_{i+1}), \\ \Rightarrow &(1 + 2\alpha)u_i - \alpha u_{i-1} - \alpha u_{i+1} = f_i, \end{aligned}$$

for all $i \in \{1, \cdots, n\}$ with $u_0 = u_1$ and $u_{n+1} = u_n$.

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Linear system with n equations and n unknowns.

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Linear system with n equations and n unknowns.

Sufficient condition?

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Sufficient condition for optimality

Definition: Convexity

We call $E : \mathbb{R}^n \to \mathbb{R}$ a convex function if for all $u, v \in C$ and all $\theta \in [0, 1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$$

We call E strictly convex, if the inequality is strict for all $\theta \in]0, 1[$, and $v \neq u$.

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Sufficient condition for optimality

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Theorem

Let $E : \mathbb{R}^n \to \mathbb{R} \in C^1(\mathbb{R}^n)$ be a convex function. Then $\nabla E(\hat{u}) = 0$ implies that $\hat{u}$ is a global minimizer of E.

Proof: Exercise sheet 2.

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Calculus of Variation

What about the continuous case?

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Calculus of Variation

What about the continuous case?

Let us start with our simple denoising example

$$u^\alpha = \underset{u \in H_0^1(\Omega)}{\operatorname{argmin}} E(u) \quad \text{with} \quad E(u) = \|u - f^\delta\|^2 + \alpha \cdot \|\partial_x u\|^2, \tag{2}$$

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Board: Let us work with the idea that

$$E(u^\alpha) \leq E(u^\alpha + \epsilon h)$$

for arbitrary numbers $\epsilon \in \mathbb{R}$ and arbitrary functions $h \in H_0^1(\Omega)$.

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Calculus of Variation

We use

Fundamental Lemma of Calculus of Variation

If a pair of continuous functions g, v on an interval $]a, b[$ meet

$$\int_a^b g(x)h(x) + v(x)\partial_x h(x)\ dx = 0$$

for all compactly supported smooth functions h on $]a, b[$, then v is differentiable, and $\partial_x v \equiv g$.

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to show that the solution to (2) meets

$$0 = u^\alpha - f - \alpha\partial_{xx}u^\alpha!$$

Continuous case and Euler-Lagrange equations

Is there a systematic concept behind this?

Euler-Lagrange Equations

Let $\rho : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with three arguments, $\rho(x, v, z)$, and consider the problem

$$\hat{u} \in \underset{u}{\operatorname{argmin}} \int_\Omega \rho(x, u(x), \nabla u(x)) \, dx.$$

Then $\hat{u}$ satisfies the **Euler-Lagrange Equations**

$$\left(\frac{d\rho}{dv} - \nabla_x \cdot \nabla_z \rho \right) (x, \hat{u}(x), \nabla \hat{u}(x)) = 0 \quad \forall x.$$

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Further considerations

Sufficient conditions for global optimality?

- Depending on the boundary conditions of u, one can get an additional condition.

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- Depending on the boundary conditions of u, one can get an additional condition.
- To go from a critical point to a global minimum one again needs convexity.

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Euler-Lagrange equations are typically too restrictive for variational problems in computer vision since they require ρ to be differentiable. A less restrictive analysis is based on **subgradients**.

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Euler-Lagrange equations are typically too restrictive for variational problems in computer vision since they require ρ to be differentiable. A less restrictive analysis is based on **subgradients**.

We will not detail this continuous analysis too much. Funny things can happen in infininte dimensions which makes the analysis more complicated.

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Discrete energy minimization only!

We will leave the continuous point-of-view for a while.

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Discrete energy minimization only!

We will leave the continuous point-of-view for a while.

Consider

$$\min_{u \in \mathbb{R}^n} E(u)$$

for $E : \mathbb{R}^n \to \mathbb{R}$.

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Consider

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for $E : \mathbb{R}^n \to \mathbb{R}$.

Strategy: Compute $\nabla E(u)$

- Can we solve $\nabla E(u) = 0$ for u directly?
- If not, apply gradient descent algorithm as presented in the following slides.

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Example for solvable $\nabla E(u) = 0$

Remember our quadratic ℓ^2-denoising problem

$$\hat{u} = \underset{u}{\operatorname{argmin}} \|u - f\|^2 + \alpha \cdot \|Du\|^2,$$

for a discrete derivative matrix D.

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We have shown / will show in the exercises that the optimality condition to such a problem is

$$0 = \hat{u} - f + \alpha D^T D\hat{u},$$
$$\Rightarrow \hat{u} = (I + \alpha D^T D)^{-1} f.$$

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Numerical methods

- Jacobi method
- Gauss-Seidel method
- Successive overrelaxation (SOR)
- Conjugate gradient (CG)

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Numerical methods

- Jacobi method
- Gauss-Seidel method
- Successive overrelaxation (SOR)
- Conjugate gradient (CG)

We won't detail the math. Use backslash or `pcg` in MATLAB. Make sure you declared $(I + \alpha D^T D)$ to be sparse!

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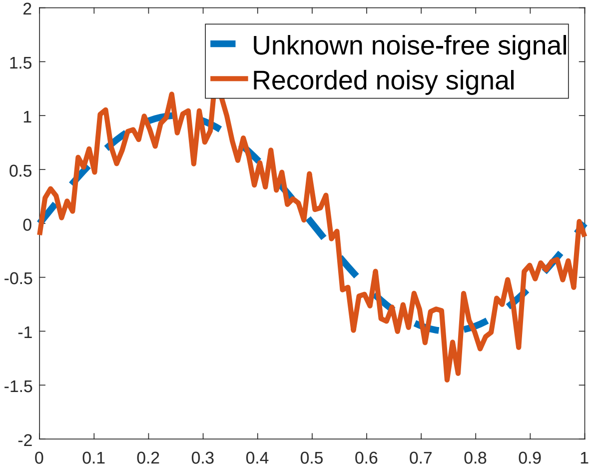
Why do we need to go beyond quadratic ℓ^2-denoising?

We got good denoising results in 1d:



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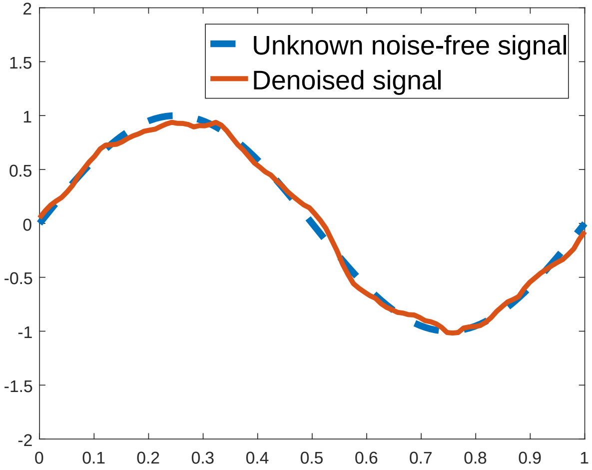

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What happens in 2d, i.e. for images?

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Why do we need to go beyond quadratic ℓ^2-denoising?

What went wrong?

Images are not continuous! Edges are extremely important for visual impression!

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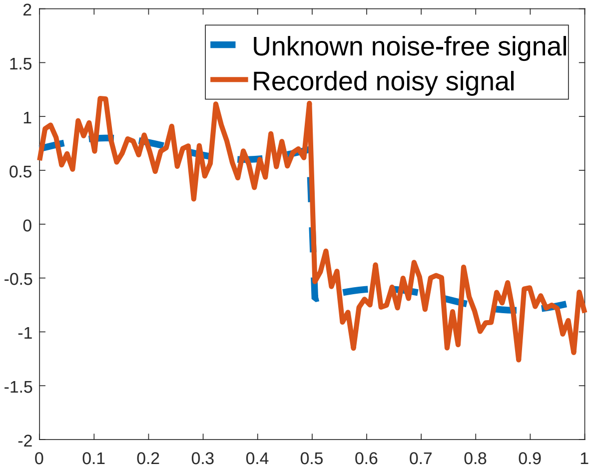

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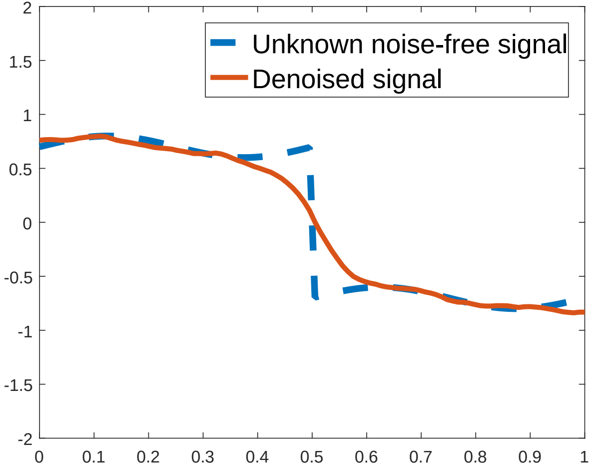

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Why does quadratic ℓ^2-denoising oversmooth edges?

Going in several small steps is cheaper than one large step!

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Why does quadratic ℓ^2-denoising oversmooth edges?

Going in several small steps is cheaper than one large step!

Assume we need to go from 0 to 1 in 10 steps. We penalize

$$E(u) = \sum_{i=1}^{10} |u_{i+1} - u_i|^2$$

and fix $u_1 = 0$, $u_{11} = 1$.

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10 equal steps:

$$E(u) = \sum_{i=1}^{10} (1/10)^2 = 10 \cdot \frac{1}{100} = \frac{1}{10}.$$

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1 big step

$$E(u) = \sum_{i=1, i\neq 5}^{10} 0^2 + 1 = 1.$$

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1 big step

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It is 10 times more expensive to take one big step!!

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Any idea on how to fix it?

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Any idea on how to fix it?

Use

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The costs will be 1 for any monotonically increasing u!!

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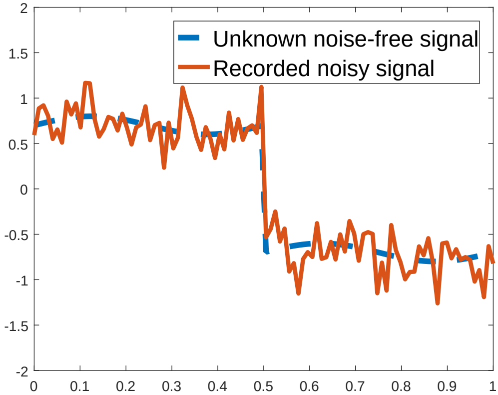

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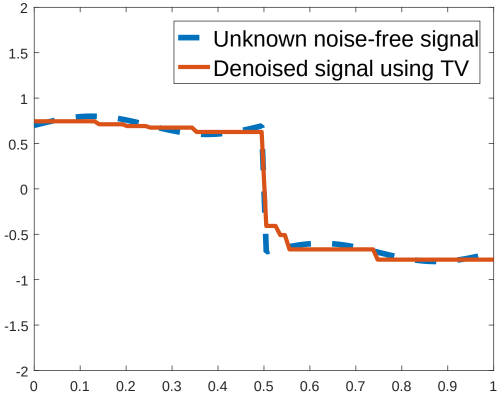

What about the optimization now?

Consider

$$\min_u \sum_{i=1}^{n} (u_i - f_i)^2 + \alpha \sum_{i=1}^{n-1} |u_{i+1} - u_i| \; = \; \min_u \|u - f\|^2 + \alpha \|Du\|_1 .$$

What is the optimality condition now?

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What is the optimality condition now?

The ℓ^1 norm is not differentiable! What can we do?

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$$\min_u \sum_{i=1}^{n} (u_i - f_i)^2 + \alpha \sum_{i=1}^{n-1} |u_{i+1} - u_i| \;=\; \min_u \|u - f\|^2 + \alpha \|Du\|_1.$$

What is the optimality condition now?

The ℓ^1 norm is not differentiable! What can we do?

While there are ways to handle the discontinuity, we will simply smooth the ℓ^1 norm by

$$S_\epsilon(d) = \sum_{i=1}^{m} \sqrt{\epsilon^2 + d_i^2},$$

and consider

$$\min_u \|u - f\|^2 + \alpha\, S_\epsilon(Du).$$

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What about the optimization now?

What is the optimality condition for

$$\min_u \|u - f\|^2 + \alpha\, S_\epsilon(Du)?$$

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What about the optimization now?

What is the optimality condition for

$$\min_u \|u - f\|^2 + \alpha\, S_\epsilon(Du)?$$

Chain rule

Let $J : \mathbb{R}^m \to \mathbb{R}$ be differentiable and $A \in \mathbb{R}^{m\times n}$. Then

$$\nabla(J \circ A)(u) = A^T \nabla J(Au).$$

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Let $J : \mathbb{R}^m \to \mathbb{R}$ be differentiable and $A \in \mathbb{R}^{m \times n}$. Then

$$\nabla(J \circ A)(u) = A^T \nabla J(Au).$$

Therefore, we find the optimality condition

$$\begin{aligned} 0 &= 2(\hat{u} - f) + \alpha D^T \nabla S(D\hat{u}) \\ &= 2(\hat{u} - f) + \alpha D^T \begin{pmatrix} \frac{(D\hat{u})_1}{\sqrt{\epsilon^2 + (D\hat{u})_1^2}} \\ \cdots \\ \cdots \\ \frac{(D\hat{u})_{n-1}}{\sqrt{\epsilon^2 + (D\hat{u})_{n-1}^2}} \end{pmatrix} \end{aligned}$$

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We will not be able to solve this in closed form!

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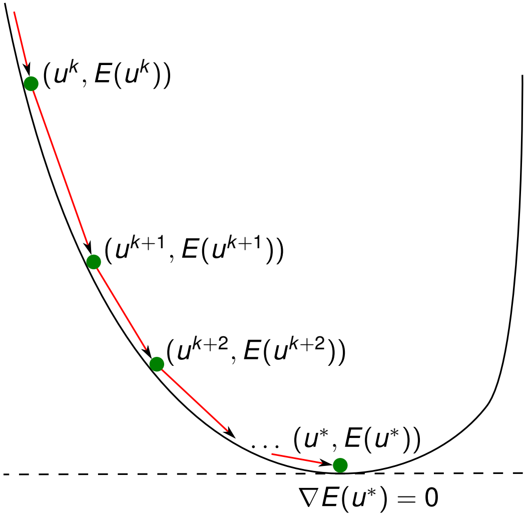
Gradient descent algorithm

Idea: For minimizing the energy E, move into the direction of steepest descent

$$u^{k+1} = u^k - \tau^k \nabla E(u^k).$$



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Gradient descent algorithm

Idea: For minimizing the energy E, move into the direction of steepest descent

$$u^{k+1} = u^k - \tau^k \nabla E(u^k).$$

How can we choose τ^k for the algorithm to converge?

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Gradient descent algorithm

Idea: For minimizing the energy E, move into the direction of steepest descent

$$u^{k+1} = u^k - \tau^k \nabla E(u^k).$$

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A convergence analysis shows that a sufficient criterion for a constant τ would be $\tau < \frac{2}{L}$ if ∇E is Lipschitz-continuous with constant L.

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We will circumvent these problems by using a backtracking line search algorithm.

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Gradient descent with backtracking line seach

Pick $\alpha \in]0, 0.5[$ and $\beta \in]0, 1[$. Iterate:

- Given an estimate u^k, compute $E(u^k)$ and $\nabla E(u^k)$.
- Initialize $\tau_k = \tau^0$.
- Find a good τ_k by:

$$
\begin{aligned}
& u^{test} = u^k - \tau_k \nabla E(u^k) \\
& \text{while } E\left(u^{test}\right) > E(u^k) - \alpha\tau_k \left\|\nabla E(u^k)\right\|^2 \\
& \qquad \tau_k \leftarrow \beta\tau_k \\
& \qquad u^{test} = u^k - \tau_k \nabla E(u^k) \\
& \text{end}
\end{aligned}
$$

- Once τ^k meets the criterion in the while-loop, update

$$u^{k+1} = u^{test}.$$

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Practical considerations:

- Guessing good values for α and β is often difficult and requires some problem-specific fine-tuning.

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- In practice one should definitely also define a maximum number of iterations.

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- Stopping criteria could be based on $\|u^k - u^{k+1}\| \leq \epsilon$ or $\|\nabla E(u^k)\| \leq \epsilon$.
- In practice one should definitely also define a maximum number of iterations.
- Allow the user to specify a starting point. Good guesses on the solution can improve the speed of convergence significantly!

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