Weekly Exercises 10

Room: H-C 7326

Wednesday, 11.01.2017, 14:15-15:45

Submission deadline: Monday, 09.01.2017, 16:00 in the lecture

Programming: email to jonas.geiping@uni-siegen.de

Theory: Radon Transformation

Recall that the *adjoint operator* A^* of a linear operator $A : X \to Y$ between Hilbert spaces X and Y meets

$$\langle Ax, y\rangle_Y = \langle x, A^*y\rangle_X \qquad \forall x \in X,\ y \in Y.$$

Exercise 1 (4 Points: Adjoint of the Radon transform)**.** Determine a linear operator $\mathcal{R}^*$ that is the adjoint operator to the Radon transform $\mathcal{R}$, i.e. it meets

$$\int_0^\pi \int_{-\infty}^\infty \mathcal{R}(u)(s,\alpha)v(s,\alpha)\ ds\ d\alpha = \int_{-\infty}^\infty \int_{-\infty}^\infty u(x,y)\mathcal{R}^*(v)(x,y)\ dx\ dy.$$

Show that the adjoint operator $\mathcal{R}^*$ is the same as the unfiltered backprojection formula

$$\frac{1}{(2\pi)^2}\int_0^\pi \int_{-\infty}^\infty \widetilde{\mathcal{R}(u)}(s,\alpha)\ e^{is\langle x,\theta(\alpha)\rangle}\ ds\ d\alpha$$

considered in the lecture, where $\theta(\alpha) = (\cos(\alpha), \sin(\alpha))^T$.

Exercise 2 (4 Points: Radon inversion formula)**.** Prove the Radon inversion formula from the lecture: For $u : \mathbb{R}^2 \to \mathbb{R}$ with u and $\hat{u}$ being absolutely integrable it holds that

$$u(x) = \frac{1}{(2\pi)^2}\int_0^\pi \int_{-\infty}^\infty \widetilde{\mathcal{R}(u)}(s,\alpha)|s|\ e^{is\langle x,\theta(\alpha)\rangle}\ ds\ d\alpha,$$

where $\theta(\alpha) = (\cos(\alpha), \sin(\alpha))^T$.

Programming: Tomographic Inversion

Exercise 3 (Bonus Exercise - 8 Points: Radon inversion with TV regularization)**.** Download the data "radonInversion.zip" from the webpage and load the contained .mat file in MATLAB. The file contains sinogram data $\mathcal{R}(u)$ in the variable `measurementMatrix` as well as the measurement angles used to simulate the data.

- Visualize the sinogram data using imagesc.

- Create a sparse matrix R that computes the Radon transform of a 16×16 pixel image using the angles stored in `measurementAngles`. What is the size of R and how many nonzeros does R have? Relate this number to the size of the input image and argue why storing R in matrix form is not feasible for reasonably sized images.
Hint: You just need to know the Radon transformation of the canonical basis elements, which MATLAB's `radon` can do for you.

- Write function handles for applying R and R^* to an image. You will need to apply the result of exercise 1 that the unfiltered backprojection is the adjoint operator. See MATLAB's 'iradon'. Due to the way MATLAB's `iradon` is implemented you need to crop the (2:end-1,2:end-1) center part in order to actually get the adjoint operator.

- Implement the variational tomographic reconstruction of the provided data using a smooth approximation of the total variation as a regularization function. *Hint: You need to be more patient than usual because applying `radon` is quite slow.*

This exercise is a Christmas-bonus-exercise and gives points which do not count to the total number of points of which 50% have to be achieved.

Merry Christmas and a Happy New Year!