Convex Optimization for Computer Vision
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Exercises: H. Bauermeister
Summer Semester 2018

Universität Siegen
Department ETI
Visual Scene Analysis

Weekly Exercises 2

Room: HF-115

Monday, 23.04.2018, 14:15-15:45,

Submission deadline: Friday, 27.04.2018, 18:00, letter box at H-A 7116

Theory: Convex Sets and Functions (20 Points)

Exercise 1 (4 Points)**.** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Prove the equivalence of the following statements:

- f is convex.
- $\text{epi}(f) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+1} : f(x) \leq y \right\}$ is convex.

Exercise 2 (4 Points)**.** Let $X \subset \mathbb{R}^n$ open and convex and let $f : X \to \mathbb{R}$ be twice continuously differentiable. This means that the function $\nabla f : X \to \mathbb{R}^n$ is continuously differentiable. Its derivative $\nabla^2 f$ is called Hessian matrix and is given by

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial f}{\partial x_n \partial x_n} \end{pmatrix}$$

Prove the equivalence of the following statements:

- f is convex.
- For all $x \in X$ the Hessian $\nabla^2 f(x)$ is positive semidefinite, i.e. $\forall\, v \in \mathbb{R}^n$: $v^\top \nabla^2 f(x) v \geq 0$.

Hints: You can use that for $x, y \in X$ it holds that f is convex if and only if

$$(y - x)^\top \nabla f(x) \leq f(y) - f(x).$$

Further recall that there are two variants of the (multidimensional) Taylor expansion:

$$f(x + tv) = f(x) + tv^\top \nabla f(x) + \frac{t^2}{2} v^\top \nabla^2 f(x) v + o(t^2)$$

with $\lim_{t \to 0} \frac{o(t^2)}{t^2} = 0$ and

$$f(x + v) = f(x) + v^\top \nabla f(x) + \frac{1}{2} v^\top \nabla^2 f(x + tv) v$$

for appropriate $t \in (0, 1)$.

Exercise 3 (4 Points)**.** Let $X \subset \mathbb{R}^n$ open and convex, $A \in \mathbb{R}^{n\times n}$ positive semidefinite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$. Show that that the quadratic form $f : X \to \mathbb{R}$ defined as

$$f(x) := \frac{1}{2}x^\top Ax + b^\top x + c,$$

is convex.

Programming: Inpainting (8 Points)

Exercise 4 (12 Points)**.** Write a MATLAB program that solves the inpainting problem

$$\min_{u\in\mathbb{R}^{n\times m}} \sum_{i,j}(u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j}\ \forall(i,j) \in I,$$

with index set I of pixels to keep. Those can be identified as the white pixels of the mask image provided on the courses homepage.
Hint: The constrained optimization problem can be reformulated so that it becomes unconstrained: Rewrite the objective as a least squares problem in terms of the unknown intensities $u_{i,j}$, $(i,j) \notin I$ using sparse linear operators: Find linear operators X, Y s.t. u can be decomposed as

$$u = X\tilde{u} + Yf$$

where $\tilde{u}$ contains only the unknown intensities. Optimize for $\tilde{u}$ instead of u. You may use MATALBs `mldivide`.