Convex Optimization for Computer Vision
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Universität Siegen
Department ETI
Visual Scene Analysis

Weekly Exercises 3

Room: HF-115

Wednessday, 17.05.2017, 12:15-14:00,

Submission deadline: Monday, 15.05.2017, 12:15, in the lecture

Theory: The Subdifferential

Exercise 1 (4 Points)**.** Let a function $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable at $x \in \mathbb{R}^n$. The *directional derivative* $\partial_v f(x)$ in the direction of v at x is defined as

$$\partial_v f(x) := \lim_{\epsilon \to 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon}.$$

Prove that

$$\partial_v f(x) = \langle \nabla f(x), v \rangle.$$

Hint: Write $\partial_v f(x)$ as the derivative of some composite function $f \circ g$.

Exercise 2 (4 Points)**.** Compute the following subdifferentials:

1. $\partial f(x)$ for $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = \|x\|_2$.
2. $\partial f(X)$ for $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ with $f(X) = \|X\|_{2,1} := \sum_{i=1}^m \|x^i\|_2$, where $x^i \in \mathbb{R}^n$ is the i-th column of X.
3. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ with $f(x) = \begin{cases} 0 & \text{if } \|x\|_2 \leq 1 \\ \infty & \text{otherwise} \end{cases}$.
   Show that $\partial f(x) = \{0\}$ if $\|x\|_2 < 1$ and $\partial f(x) = \{\alpha x \mid \alpha \geq 0\}$ if $\|x\|_2 = 1$.

Exercise 3 (4 Points)**.** Let $f \in \mathbb{R}^n$. Show that the solution u of the convex optimization problem

$$\arg\min_{u \in \mathbb{R}^n} \frac{1}{2\lambda} \|u - f\|^2 + \|u\|_1,$$

is given as

$$u \in \mathbb{R}^n, \quad u_i := \begin{cases} f_i + \lambda & \text{if } f_i < -\lambda \\ 0 & \text{if } f_i \in [-\lambda, \lambda] \\ f_i - \lambda & \text{if } f_i > \lambda. \end{cases}$$

This is an important formula known as *soft thresholding* or *shrinkage.*

Hint: Note that the above optimization problem is decoupled in the sense that one can look for the individual entries u_i of the optimal u separately.

Programming - Wavelet Shrinkage

Exercise 4 (4 points)**.** Use the results from the last exercise to implement a denoising algorithm with wavelet shrinkage. Compute a wavelet decomposition Wf of an image f, minimize

$$\arg\min_{c\in\mathbb{R}^n} \frac{1}{2\lambda}\|c - Wf\|^2 + \|c\|_1,$$

for some well-chosen λ and recompose the denoised image with $u^* = W^{-1}c^*$.
Test your program with the image `giraffe.jpg`, converted to gray-scale with some added noise.
You can use the MATLAB functions

```
[c,S] = wavedec2(f,6,'db5')
[u] = waverec2(c,S,'db5')
```

as a black-box to transform some 2d image f into coefficients c and back.

Remark: `'db5'` is the type of wavelet, that is used for the transformation (here a *Daubechies* with 5 vanishing moments), and the 6 stands for 6 scales of decomposition. The specifics of the decomposition are saved in the matrix `S` which has to be used for the reconstruction.