

Weekly Exercises 3

Room: HF-115

Thursday, 23.05.2019, 8:30-10:00,

Submission deadline: Wednesday, 22.05.2019, 18:00, letter box at H-A 7116/email

Theory: The Subdifferential

Exercise 1 (4 Points). Let a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable at $x \in \mathbb{R}^n$. The *directional derivative* $\partial_v f(x)$ in the direction of v at x is defined as

$$\partial_v f(x) := \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon}.$$

Prove that

$$\partial_v f(x) = \langle \nabla f(x), v \rangle.$$

Hint: Write $\partial_v f(x)$ as the derivative of some composite function $f \circ g$.

Exercise 2 (4 Points). Compute the following subdifferentials:

1. $\partial f(x)$ for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $f(x) = \|x\|_2$.
2. $\partial f(X)$ for $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ with $f(X) = \|X\|_{2,1} := \sum_{i=1}^m \|x^i\|_2$, where $x^i \in \mathbb{R}^n$ is the i -th column of X .
3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ with $f(x) = \begin{cases} 0 & \text{if } \|x\|_2 \leq 1 \\ \infty & \text{otherwise} \end{cases}$.

Show that $\partial f(x) = \{0\}$ if $\|x\|_2 < 1$ and $\partial f(x) = \{\alpha x \mid \alpha \geq 0\}$ if $\|x\|_2 = 1$.

Exercise 3 (4 Points). Let $W \in \mathbb{R}^{n \times n}$ be orthogonal and $\lambda > 0$. Show that the minimization problem

$$\arg \min_{u \in \mathbb{R}^n} \frac{1}{2} \|u - f\|^2 + \lambda \|Wu\|_1$$

is equivalent to the formulation used in the programming exercise

$$\arg \min_{c \in \mathbb{R}^n} \frac{1}{2\lambda} \|c - Wf\|^2 + \|c\|_1.$$

Hint. A matrix $W \in \mathbb{R}^{n \times n}$ is orthogonal, iff it holds that $W^T W = W W^T = I$. Show that the ℓ^2 -norm is invariant with respect to multiplications with orthogonal matrices, i.e. $\|Wu\| = \|u\|$ for all $u \in \mathbb{R}^n$.

Programming - Wavelet Shrinkage

Exercise 4 (4 points). Use the results from the last exercise to implement a denoising algorithm with wavelet shrinkage. Compute a wavelet decomposition Wf of an image f , minimize

$$\arg \min_{c \in \mathbb{R}^n} \frac{1}{2\lambda} \|c - Wf\|^2 + \|c\|_1,$$

for some well-chosen λ and recompose the denoised image with $u^* = W^{-1}c^*$.

Test your program with the image `giraffe.jpg`, converted to gray-scale with some added noise.

You can use the MATLAB functions

```
[c,S] = wavedec2(f,6,'db5')
```

```
[u] = waverec2(c,S,'db5')
```

as a black-box to transform some 2d image f into coefficients c and back.

Remark: 'db5' is the type of wavelet, that is used for the transformation (here a *Daubechies* with 5 vanishing moments), and the 6 stands for 6 scales of decomposition. The specifics of the decomposition are saved in the matrix S which has to be used for the reconstruction.