Variational Methods for Computer Vision
Lecture: M. Möller
Exercises: J. Geiping
Winter Semester 16/17

Visual Scene Analysis
Institute for Computer Science
University of Siegen

Weekly Exercises 4

Room: H-C 7326
Wednesday, 16.11.2016, 14:15-15:45
Submission deadline: Monday, 14.11.2016, 16:00 in the lecture
Programming: email to jonas.geiping@uni-siegen.de

Theory

Exercise 1 (2 Points)**.** Let

$$W = \begin{pmatrix} A & B \\ B^T & B^T A^{-1} B \end{pmatrix}$$

for $W \in \mathbb{R}^{N\times N}$, $A \in \mathbb{R}^{r\times r}$, and $B \in \mathbb{R}^{r\times N-r}$. Prove that $\text{rank}(W) \leq r$ by showing the identity

$$W = \bar{U}\ \Sigma\ \bar{U}^T, \qquad \text{for} \quad \bar{U} = \begin{pmatrix} U \\ B^T U \Sigma^{-1} \end{pmatrix} \in \mathbb{R}^{N\times r}$$

where $A = U\Sigma U^T$ is the eigendecomposition of A.

It can be shown that this W with $\text{rank}(W) \leq r$ has an eigendecomposition $W = VDV^T$ for some $V \in \mathbb{R}^{N\times r}$ and D being diagonal [1], but you can skip this step.

Exercise 2 (4 points)**.** Let $W \in \mathbb{R}^{N\times N}$ be a rank r matrix with eigendecomposition $W = VDV^T$. Show that the solution to

$$\hat{u} = \text{argmin}_u \frac{1}{2}\|u - f\|^2 + \frac{\alpha}{2}\langle u, (I - W)u\rangle$$

is $\hat{u} = f + V\tilde{D}V^T f$, where $\tilde{D}$ is a diagonal matrix with $\tilde{D}_{ii} = \frac{1}{1+\alpha+\alpha\sigma_i} - \frac{1}{1+\alpha}$ and σ_i are the diagonal entries of D.
Start with the optimality condition and try to compute $V^T\hat{u}$ first, then deduct the solution.
Hint: Consider the decomposition $UU^T x + (I - UU^T)x = 0$ for your optimality condition.

[1] See Fowlkes et al. *Spectral Grouping Using the Nyström method*, (2004)

Understanding Code

Exercise 3 (4 points)**.** Download the functions "IntegralNLM.m", "IntegralImage.m", and "ImShift.m" from the course's website. Answer the following questions:

- What does "ImShift.m" do?
- What does "IntegralImage.m" do?
- State a mathematical formula for the (i, j)-th entry of the variable *ssd* in IntegralNLM.m.
- Based on the previous exercise, state a formula for *PatchDist*.

Programming

Exercise 4. Load the image *bikes* provided on the course's website in Matlab and create a noisy version by adding Gaussian noise (*help randn*). Use the downloaded functions from the previous exercise to generate a similarity matrix W of the noisy image.

- Perform nonlocal means denoising with the similarity matrix W
- Compute a diagonal matrix D with

  $$d_i = \sum_j W_{i,j}$$

  on the diagonal. Perform nonlocal regularization with

  $$R(u) = \frac{\alpha}{2}\langle u, Lu \rangle$$

  for $L = D - W$ and for $L = I - D^{-1/2} W D^{-1/2}$. Tune the values of α.
- Run your TV denoising code from the previous exercise sheet.

Which denoising algorithm yields the best results?