Weekly Exercises 4

Room: HF-115

Wednessday, 24.05.2017, 12:15-14:00,

Submission deadline: Monday, 22.05.2017, 12:15, in the lecture

Theory: Subdifferentials, Lipschitz continuity, and fixed point iterations

Exercise 1 (4 Points). We call a function $E : \mathbb{R}^n \to \mathbb{R}$ *absolutely one-homogeneous* if

$$E(\alpha u) = |\alpha| E(u)$$

holds for all $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Prove that

$$\partial E(u) = \{p \in \mathbb{R}^n \mid \langle p, u \rangle = E(u), \quad E(v) \geq \langle p, v \rangle \ \forall v \in \mathbb{R}^n\}.$$

Exercise 2 (4 Points). Find examples for the following functions and explain why your example is correct:

- A continuously differentiable convex function that is not L-smooth.
- A Lipschitz continuous function that is not a contraction.
- A function that is not differentiable, but Lipschitz continuous.
- A convex L-smooth function E and a step size τ for which G defined by $G(u) = u - \tau \nabla E$ is not a non-expansive function.

Exercise 3 (4 Points). Show that for any $a, b \in \mathbb{R}^n$, $\theta \in \mathbb{R}$ it holds that

$$\|(1-\theta)a + \theta b\|^2 = (1-\theta)\|a\|^2 + \theta\|b\|^2 - \theta(1-\theta)\|a-b\|^2$$

Programming: Unstable step sizes

Exercise 4 (4 Points). Consider minimizing the energy

$$E(u) = \frac{1}{2}\|u - f\|^2 + \alpha\|Du\|_1.$$

It is easy to see that

$$p(u) := u - f + \alpha D^T \mathrm{sign}_0(Du) \in \partial E(u),$$

for $\operatorname{sign}_0(0) = 0$.

Implement a gradient descent iteration of the form

$$u^{k+1} = u^k - \tau p(u^k),$$

vary the step size τ, and measure how close your iterates are to being optimal by plotting $\|p(u^k)\|_2$. What do you observe? Depending on the step size, there should be (at least) two different types of behaviors.