Variational Methods for Computer Vision
Lecture: M. Möller
Exercises: J. Geiping
Winter Semester 16/17

Visual Scene Analysis
Institute for Computer Science
University of Siegen

Weekly Exercises 5

Room: H-C 7326

Wednesday, 23.11.2016, 14:15-15:45

Submission deadline: Monday, 21.11.2016, 16:00 in the lecture

Programming: email to jonas.geiping@uni-siegen.de

Theory

Exercise 1 (4 Points)**.** Let f, and g be functions in $L^1(\mathbb{R})$. The latter means that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are absolutely integrable, i.e.

$$\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)| \; dx < \infty, \quad \|g\|_{L^1} = \int_{\mathbb{R}} |g(x)| \; dx < \infty.$$

Let

$$\mathcal{F}(f)(z) := \int_{\mathbb{R}} f(x) \exp(-2\pi i x z) \; dx$$

denote the *Fourier transformation* of f.

Consider the convolution

$$c(x) = (f * g)(x) = \int_{\mathbb{R}} f(y) g(x-y) \; dy.$$

- Show that c is also absolutely integrable by proving $\|c\|_1 \leq \|f\|_1 \|g\|_1$.
- Prove the following theorem about the relation between the convolution and the Fourier transformation:

$$\mathcal{F}(c) = \mathcal{F}(f) \cdot \mathcal{F}(g).$$

Hint: For absolutely integrable functions you may exchange the order of integration (Fubini's theorem).

Exercise 2 (4 Points)**.** Let $f, g \in \mathbb{R}^n$ be a discrete signal. Let

$$F_k = \sum_{l=0}^{n-1} f_l \exp(-2\pi i k l/n), \quad G_k = \sum_{m=0}^{n-1} g_m \exp(-2\pi i k m/n), \quad k \in \mathbb{Z}$$

denote the (infinitely many, but n-periodic) coefficients of the *discrete Fourier transforms* of f and g respectively.

Consider the discrete circular convolution with coefficients

$$c_k := \sum_{l=0}^{n-1} f_l \; g_{((k-l))_n},$$

where $g_{((k-l))_n}$ denotes the n-periodic extension of g (so $(k-l) \bmod n$ if you think back to your discrete mathematics lectures.)

Prove that the discrete circular convolution theorem

$$C_k = F_k \cdot G_k$$

holds, where C_k is the Fourier transform of the circular convolution of f and g.

Programming

Exercise 3 (4 Points)**.** Use the formula

$$\hat{g} = \mathcal{FFT}^{-1}(\mathcal{FFT}(f) \cdot \mathcal{FFT}(k))$$

to generate a blurry image with k being a suitable blur kernel. Here $\mathcal{FFT}$ denotes a fast implementation of the discrete Fourier transform and $\mathcal{FFT}^{-1}$ its corresponding inverse transform.

Verify that

$$\hat{u} = \mathcal{FFT}^{-1}(\mathcal{FFT}(f)/\mathcal{FFT}(k))$$

returns the true image without a blur.

Now compute

$$\hat{u} = \mathcal{FFT}^{-1}(\mathcal{FFT}(f + \sigma \cdot \text{noise})/\mathcal{FFT}(k))$$

for a tiny $\sigma > 0$. What do you observe?

Exercise 4 (4 Points)**.** Implement

$$\min_u \frac{1}{2}\|Au - f\|^2 + \alpha R(u)$$

for a regularization R of your choice and A being the matrix representation of the blur kernel you used in the previous exercise. Are you able to stabilize the inverse process?