Variational Methods for Computer Vision
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Winter Semester 17/18

Visual Scene Analysis
Institute for Computer Science
University of Siegen

Weekly Exercises 8

Room: H-C 6336

Friday, 15.12.2017, 14:15-15:45

Submission deadline: Tuesday, 12.12.2017, 14:15 in Room H-C 6336

Programmming: Email your solution to `jonas.geiping@uni-siegen.de`

Theory

Exercise 1 (4 Points: Derivatives with the Frobinius norm)**.** Consider the energy function

$$E(U) = \|AUB - F\|_F^2$$

where $A \in \mathbb{R}^{n\times m}$, $U \in R^{m\times p}$, $B \in \mathbb{R}^{p\times r}$, and $F \in \mathbb{R}^{n\times r}$ are matrices. The norm $\|\cdot\|_F$ denotes the Frobenius norm, i.e. $\|Z\|_F^2 = \sum_{i,j}(Z_{i,j})^2$.

Determine $\nabla E(U)$ in matrix form.

Possible hint: You may want to use the Kronecker product:

$$S \otimes T = \begin{pmatrix} S_{1,1}T & \cdots & S_{1,n}T \\ \vdots & \ddots & \vdots \\ S_{m,1}T & \cdot & S_{m,n}T \end{pmatrix}$$

and redce the problem to a vectorized problem (for which you know exactly how to differentiate it), derive a rule for $(S\otimes T)^T$*, and undo the vectorization.*

Exercise 2 (4 Points: Rank-1 Approximation)**.** Consider the problem

$$\hat{U} = \arg\min_{U\in\mathbb{R}^{n\times m}} \|U - F\|_F^2 \qquad \text{s.t. } \operatorname{rank}(U) = 1,$$

for $F \in \mathbb{R}^{n\times m}$. Our goal is to derive an explicit formula for $\hat{U}$.

- Show that the Frobenius norm is invariant under multiplication with orthonormal matrices, that is show that $||TF||_F = ||FS||_F = ||F||_F$ holds for arbitrary unitary matrices $T \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{m\times m}$.
- Assume that $F = V\Sigma Z^T$ is the (full) singular value decomposition of F. Use the previous property along with a variable substitution to derive an equivalent problem
$$\hat{D} = \arg\min_{D} \|D - \Sigma\|_F \qquad \text{s.t. } \operatorname{rank}(D) = 1.$$
What is the relation between $\hat{D}$ and $\hat{U}$?

- State a formula for $\hat{D}$ and $\hat{U}$[1].

Hint: $||\cdot||_F$ *is again the Frobenius norm, it can also be written as square root of the trace of* $A^T A$*:* $||A||_F = \sqrt{tr(A^T A)}$

Programming

Exercise 3 (4 Points - Freeing the Lion)**.** The supplementary material for this exercise contains an image of a lion f and a mask m. Solve the inverse problem of inpainting,

$$\min_u ||Au - f||^2 + \alpha R(u)$$

with the inpainting operator A given by $(Au)_i = \begin{cases} u_i & \text{if } m_i = 0 \\ 0 & \text{if } m_i = 1 \end{cases}$.

Use a regularization of your choice.

[1] Although it seems quite clear, you may use without a proof that the optimal matrix $\hat{D}$ is diagonal. If you are interested in details consider for instance Mirsky, 1958, "Symmetric Gauge Functions and Unitarily Invariant Norms", Theorem 5.