Variational Methods for Computer Vision
Lecture: M. Möller
Exercises: J. Geiping
Winter Semester 16/17

Visual Scene Analysis
Institute for Computer Science
University of Siegen

Weekly Exercises 9

Room: H-C 7326

Wednesday, 21.12.2016, 14:15-15:45

Submission deadline: Monday, 19.12.2016, 16:00 in the lecture

Programming: email to jonas.geiping@uni-siegen.de

Theory - Sparse Recovery

In the lecture we discussed that penalizing with ℓ^1 norm induces sparsity of the solution. The next two exercise give further insight on why this is the case.

Exercise 1 (4 Points: Convex Relaxation)**.** Consider the l^0 'norm' of some u on a box constraint, i.e. a fixed interval $[-a, a]$:

$$E(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{if } u \in [-a, a] \\ \infty & \text{else.} \end{cases} \tag{1}$$

Find the convex relaxation of E by computing its biconjugate E^{**}.

Exercise 2 (4 Points: Geometric Interpretation)**.** In this exercise we will visualize the difference between l^1 and l^2 minimization geometrically.

- Make a coordinate system for $x = (x_1, x_2)$ and draw the norm balls $||x||_1 \leq 1$, $||x||_2 \leq 1$ and $||x||_\infty \leq 1$.
- Draw a new coordinate system as well as the solution to the linear equation
$$x_1 + 2x_2 = 5 \tag{2}$$
- Draw the smallest l^1-norm ball into this coordinate system that exactly touches the solution. This is the l^1-minimizing solution to the underdetermined linear equation (Why ?)
- Do the same (in a second color) for the l^2-norm ball.
- *Analyze your findings:* Which solution is more sparse, i.e. has less non-zero entries ? Consider a general linear equation
$$ax_1 + bx_2 = c \tag{3}$$
for which values of a, b and c does the l^1 minimizing solution have only one entry ? For which values of a, b and c does the l^2 minimizing solution have only one entry ?

Exercise 3 (4 Points: Sparse Recovery with OMP)**.** As an alternative to ℓ^1 minimization, we learned about the (greedy) orthogonal matching pursuit (OMP) algorithm for sparse recovery. In this exercise we will learn about a criterion that guarantees the greedy strategy to be provably successful.

Assume we have a dictionary D with normalized columns, i.e. $\|D_{:,i}\|_2 = 1$ for all i. Assume that our data f has a 2-sparse representation, i.e. there exist $\alpha, \beta \in \mathbb{R}$, k, $l \in \mathbb{N}$, such that $f = \alpha D_{:,k} + \beta D_{:,l}$. Further assume that the dictionary is *incoherent* in such a way that

$$\mu := \max_{i \neq j} |\langle D_{:,i}, D_{:,j} \rangle| < 1/3.$$

Prove that two iterations of the OMP algorithm yield a solution u^2 which has two nonzero entries α and β at positions k and l respectively.

Programming

Exercise 4 (4 Points: Denoising with a learned dictionary)**.** For this exercise we will implement the orthogonal matching pursuit (OMP) algorithm to approximately solve

$$\min_u ||f - Du||_2^2 \quad \text{s.t. } |u|_0 \leq s \tag{4}$$

for a given dictionary D, noisy data patches f and a sparsity level s.

- Download the supplementary material for this exercise and visualize the given 256 learned 8x8 patches. *Hint: the MATLAB function 'col2im' might help.*
- Implement the OMP algorithm given in the lecture.
- Load a test image, add noise (e.g. *cameraman.tif* found in every MATLAB path) and write a denoising by the OMP algorithm. Try to find a sensible choice of s. The MATLAB function *im2col* might help you to transfer the image into a patch representation. Use distinct patches at first.
- Modify your denoising algorithm from distinct patches to 'sliding' patches to reduce artifacts.
- Modify your denoising algorithm by removing the mean from each noisy data patch before OMP and adding it afterwards.

P.S: For the exercise on Wednesday 14.12, bring a laptop for the live exercise, if you can.