Convex Optimization for Computer Vision
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Universität Siegen
Department ETI
Visual Scene Analysis

Weekly Exercises 9

Room: HA-7116

Wednessday, 28.06.2017, 12:15-14:00,

Submission deadline: Monday, 26.06.2017, 12:15, in the lecture

Theory: Convex conjugation part 2

Exercise 1 (2 Points)**.** Compute the convex conjugates of the following functions:

$$\delta_{\geq 0} : \mathbb{R}^{n\times m\times k} \to \mathbb{R}\cup\{\infty\}$$
$$u \mapsto \delta_{\geq 0}(u) = \begin{cases} 0 & \text{if } u_{ijl} \geq 0, \quad \forall (i,j,l) \\ \infty & \text{otherwise.} \end{cases}$$

and

$$\delta_{\Sigma} : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$$
$$v \mapsto \delta_{=1}(v) = \begin{cases} 0 & \text{if } v_{ij} = 1, \quad \forall (i,j) \\ \infty & \text{otherwise.} \end{cases}$$

Theory: Preparing PDHG

Exercise 2 (4 Points)**.** Consider the problem

$$\min_u G(u) + F_1(K_1 u) + F_2(K_2 u).$$

Why are the following updates a possible PDHG implementation of the above minimization problem (assuming all technical conditions are met)?

$$\begin{aligned} p_1^{k+1} &= \operatorname{prox}_{\sigma F_1^*}(p_1^k + \sigma K_1 \bar{u}^k) \\ p_2^{k+1} &= \operatorname{prox}_{\sigma F_2^*}(p_2^k + \sigma K_2 \bar{u}^k) \\ u^{k+1} &= \operatorname{prox}_{\tau G}(u^k - \tau K_1^T p_1^{k+1} - \tau K_2^T p_2^{k+1}) \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k \end{aligned}$$

Hint: Exploit that $F_1^{**} = F_1$ and $F_2^{**} = F_2$!

Programming: The primal-dual hybrid gradient method

Exercise 3 (10 Points)**.** In this exercise your task is to compute a piecewise constant, cartoonish looking approximation of the input image. This can be done as follows: We begin selecting k different colors (e.g. $k = 4$), $\{c_1, c_2, \dots c_k\}$ that are most present in the image, for example $c_1 =$ red, $c_2 =$ green, $c_3 =$ blue and $c_4 =$ yellow. We then segment the image into k disjoint regions, so that the overall boundary length is short and at the same time the pixels in the j-th region are close to the j-th color. For an $n \times m$ image g, one can solve the following optimization problem

$$\min_{u \in \mathbb{R}^{n \times m \times k}} \delta_{\geq 0}(u) + \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{l=1}^{k} u_{ijl} f_{ijl} + \alpha \|Du\|_1 + \delta_{=1}(Su),$$

where $f_{ijl} = \|g_{i,j,:} - c_l\|_2$ is the Euclidean distance of the pixel (i,j) color to the color c_l. $D : \mathbb{R}^{n \times m \times k} \to \mathbb{R}^{2nmk}$ is a linear operator that stacks the results of applying a finite difference approximation of the gradient to each channel of u. $S : \mathbb{R}^{n \times m \times k} \to \mathbb{R}^{n \times m}$ sums the values in the third input dimension, i.e. $(S(u))_{i,j} = \sum_{l=1}^{k} u_{ijl}$.

Let $\tilde{u}$ be a minimizer of the problem above. Your final solution $\bar{u} \in \mathbb{R}^{n \times m \times 3}$ is then given as

$$\bar{u}_{i,j,:} := c_m \quad \text{where } m := \arg\max_{l} \tilde{u}_{ijl}.$$

Transform the problem above into a saddle point problem, i.e. identify F, G and the linear operator K, derive the proximal operators and solve it with PDHG.
Hints:

- Vectorize your problem, i.e. $u \in \mathbb{R}^{knm}$. Then the term $\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{l=1}^{k} u_{ijl} f_{ijl}$ is just a scalar product $\langle u, f \rangle$.

- Formalize your algorithm using the result of exercise 2 with

$$F_1(K_1 u) = \|\alpha D u\|_1, \qquad F_2(K_2 u) = \delta_{=1}(Su).$$

  Use the results of exercise 1 for the conjugates of F_1 and F_2.

- Remember the Kronecker product to construct matrices for the involved linear operators. Reuse your old code.

- You may use MATLAB `kmeans` to find k representative colors of the input image.

- Contact me and ask for help if certain steps are unclear!