Variational Methods for Computer Vision
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Exercises: J. Geiping
Winter Semester 17/18

Visual Scene Analysis
Institute for Computer Science
University of Siegen

Weekly Exercises 9

Room: H-C 6336

Friday, 22.12.2017, 14:15-15:45

Submission deadline: Tuesday, 19.12.2017, 14:15 in Room H-C 6336

Programmming: Email your solution to `jonas.geiping@uni-siegen.de`

Theory - Sparse Recovery

In the lecture, as well as in exercise 3, we use the OMP algorithm to solve ℓ^0-penalized problems [1]. Another classic solution to ℓ^0-problems is to "relax" the penalty to a convex function. We will compute this relaxation (remember we discussed convex relaxations in the lecture during image demosaicking) now and check its properties.

Exercise 1 (4 Points: Convex Relaxation)**.** Consider the l^0 'norm' of some u on a box constraint, i.e. a fixed interval $[-a, a]$:

$$E(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{if } u \in [-a, a] \\ \infty & \text{else.} \end{cases} \tag{1}$$

Find the convex relaxation of E by computing its biconjugate E^{**}.

Hint: First compute the convex conjugate, then compute the convex conjugate of your convex conjugate. Use case differentiations. There are actually a few different cases you have to consider for each conjugate.

Exercise 2 (4 Points: Geometric Interpretation)**.** In this exercise we will visualize the difference between l^0,l^1 and l^2 minimization geometrically.

- Make a coordinate system for $x = (x_1, x_2)$ and draw the norm balls $||x||_1 \leq 1$, $||x||_2 \leq 1$ and $||x||_\infty \leq 1$. Then mark all points where $|x|_0 \leq 1$.
- Draw a new coordinate system as well as the solution to the linear equation

$$x_1 + 2x_2 = 5 \tag{2}$$

- Draw the smallest l^1-norm ball into this coordinate system that exactly touches the solution. This is the l^1-minimizing solution to the underdetermined linear equation (Why ?)

[1] Reminder: $\ell^0(u) = \#(u_i \neq 0) = \{\text{number of nonzeros elements in vector } u\}$

- Do the same (in a second and third color) for the l^2-norm ball and the l^0 markings.

- *Analyze your findings:* Which solution is more sparse, i.e. has less non-zero entries? Consider a general linear equation

$$ax_1 + bx_2 = c \tag{3}$$

for which values of a, b and c does the l^1 minimizing solution have only one entry ? For which values of a, b and c does the l^2 minimizing solution have only one entry? Which one is more faithful to the l^0 solution?

Programming

Exercise 3 (8 Points: Denoising with a learned dictionary)**.** For this exercise we will implement the orthogonal matching pursuit (OMP) algorithm to approximately solve

$$\min_u ||f - Du||_2^2 \quad \text{s.t. } |u|_0 \leq s \tag{4}$$

for a given dictionary D, noisy data patches f and a sparsity level s.

- Download the supplementary material for this exercise and visualize the given 256 learned 8x8 patches. *Hint: the MATLAB function 'col2im' might help.*

- Implement the OMP algorithm given in the lecture.

- Load a test image, add noise (e.g. *peppers*, reduce its size if necessary for reasonably fast computation)) and write the OMP denoising algorithm. Try to find a sensible choice of s. The MATLAB function *im2col* might help you to transfer the image into a patch representation. Use distinct patches at first.

- Modify your denoising algorithm from distinct patches to 'sliding' patches to reduce artifacts.

- Modify your denoising algorithm by removing the mean from each noisy data patch before OMP and adding it afterwards.